

# GEOMETRY AND CLASSIFICATION OF SOLUTIONS OF THE CLASSICAL DYNAMICAL YANG-BAXTER EQUATION

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**ABSTRACT.** The classical Yang-Baxter equation (CYBE) is an algebraic equation central in the theory of integrable systems. Its solutions were classified by Belavin and Drinfeld. Quantization of CYBE led to the theory of quantum groups. A geometric interpretation of CYBE was given by Drinfeld and gave rise to the theory of Poisson-Lie groups.

The classical dynamical Yang-Baxter equation (CDYBE) is an important differential equation analogous to CYBE and introduced by Felder as the consistency condition for the differential Knizhnik-Zamolodchikov-Bernard equations for correlation functions in conformal field theory on tori. Quantization of CDYBE allowed Felder to introduce an interesting elliptic analog of quantum groups. It becomes clear that numerous important notions and results connected with CYBE have dynamical analogs.

In this paper we classify solutions to CDYBE and give geometric interpretation to CDYBE. The classification and interpretation are remarkably analogous to the Belavin-Drinfeld picture.

## 0. INTRODUCTION

**0.1.** In 1984 Knizhnik and Zamolodchikov showed that correlation functions of conformal blocks on  $\mathbb{P}^1$  for the Wess-Zumino-Witten (WZW) conformal field theory for a simple Lie algebra  $\mathfrak{g}$  satisfy the differential equations

$$(1) \quad \kappa \frac{\partial F}{\partial z_i} = \sum_{j \neq i} \frac{\Omega^{ij}}{z_i - z_j} F, \quad i = 1, \dots, N,$$

where  $F$  is an analytic function of  $N$  complex variables  $z_1, \dots, z_N$  with values in  $V_1 \otimes \dots \otimes V_N$ ,  $V_1, \dots, V_N$  are representations of  $\mathfrak{g}$ ,  $\Omega^{ij}$  is the Casimir operator  $\Omega \in (S^2 \mathfrak{g})^{\mathfrak{g}}$  acting in the  $i$ -th and the  $j$ -th component, and  $\kappa$  is a complex number. Equations (1) are called the Knizhnik-Zamolodchikov (KZ) equations. They play an important role in representation theory and mathematical physics.

Later Cherednik [C] generalized the KZ equations: he considered differential equations of the form

$$(2) \quad \kappa \frac{\partial F}{\partial z_i} = \sum_{j \neq i} r^{ij}(z_i - z_j) F,$$

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where  $r(z)$  is a meromorphic function on  $\mathbb{C}$  with values in  $\mathfrak{g} \otimes \mathfrak{g}$  satisfying the unitarity condition  $r(z) = -r^{21}(-z)$ . Cherednik pointed out that system (2) is consistent iff  $r(z)$  satisfies the classical Yang-Baxter equation (CYBE):

$$(3) \quad [r^{12}(z_1 - z_2), r^{13}(z_1 - z_3)] + [r^{12}(z_1 - z_2), r^{23}(z_2 - z_3)] + [r^{13}(z_1 - z_3), r^{23}(z_2 - z_3)] = 0.$$

In particular, for the simplest Yang's solution of (3),  $r(z) = \frac{\Omega}{z}$ , we get the KZ equations.

The geometric meaning of equation (3) was found by Drinfeld [Dr]. Namely, he showed that any solution  $r$  of (3), independent of  $z$ , and satisfying the condition  $r + r^{21} \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ , defines a natural Poisson-Lie structure on a Lie group  $G$  whose Lie algebra is  $\mathfrak{g}$ . He also found that  $z$ -dependent solutions of (3) define analogous structures on the loop group  $LG$ .

If  $\mathfrak{g}$  is a simple Lie algebra, then equation (3) has many interesting solutions, both  $z$ -independent and  $z$ -dependent. These solutions, satisfying an additional nondegeneracy condition, were classified by Belavin and Drinfeld [BD1],[BD2]. In the  $z$ -dependent case, three types of solutions were found: rational, trigonometric, and elliptic (in terms of  $z$ ).

**0.2.** In [B] Bernard found that correlation functions for WZW conformal blocks on an *elliptic curve* satisfy the differential equations

$$(4) \quad \kappa \frac{\partial F}{\partial z_i} = \sum_{j \neq i} r_{KZB}^{ij}(\lambda, z_i - z_j) F + \sum_l x_l^i \frac{\partial F}{\partial x_l},$$

where  $\lambda \in \mathfrak{h}^*$ ,  $\mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}$ ,  $F$  is an analytic function of complex variables  $z_1, \dots, z_N$  and of a point  $\lambda$ , with values in  $(V_1 \otimes \dots \otimes V_N)^{\mathfrak{h}}$ ,  $r_{KZB}(z, \lambda)$  is a particular meromorphic (in fact, elliptic) function with values in  $(\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{h}}$ , and  $\{x_l\}$  is a basis of  $\mathfrak{h}$ , which is also regarded as a linear system of coordinates on  $\mathfrak{h}^*$ . Equations (4) are called the Knizhnik-Zamolodchikov-Bernard equations.

Equations (4) are not of type (2), since they contain derivatives with respect to  $x_l$  on the right hand side. Therefore, it is not surprising that  $r_{KZB}$  does not satisfy the classical Yang-Baxter equation. However, as was pointed out by Felder [F1],  $r_{KZB}$  satisfies a generalization of the classical Yang-Baxter equation:

$$(5) \quad \sum_l x_l^{(1)} \frac{\partial r^{23}(z_2 - z_3)}{\partial x_l} + \sum_l x_l^{(2)} \frac{\partial r^{31}(z_1 - z_3)}{\partial x_l} + \sum_l x_l^{(3)} \frac{\partial r^{12}(z_1 - z_2)}{\partial x_l} + [r^{12}(z_1 - z_2), r^{13}(z_1 - z_3)] + [r^{12}(z_1 - z_2), r^{23}(z_2 - z_3)] + [r^{13}(z_1 - z_3), r^{23}(z_2 - z_3)] = 0.$$

Moreover, this equation is a necessary and sufficient condition for (4) to be consistent (for an arbitrary meromorphic function  $r(\lambda, z)$  satisfying the unitarity condition  $r(\lambda, z) = -r^{21}(\lambda, -z)$ ). This equation is called the *classical dynamical Yang-Baxter equation* (we will abbreviate it as CDYBE, or CDYB equation). The word “dynamical” refers to the fact that (5) is a differential rather than an algebraic equation, so it reminds of a dynamical system.

This paper has two goals:

1. To exhibit the geometric meaning of CDYBE, similarly to Drinfeld's interpretation of CYBE within the framework of the theory of Poisson-Lie groups.
2. To classify solutions of CDYBE for simple Lie algebras and Kac-Moody algebras (over  $\mathbb{C}$ ).

**0.3.** The first goal is attained in Chapters 1 and 2. Namely, there we consider solutions of (5), for any pair of finite-dimensional Lie algebras  $\mathfrak{h} \subset \mathfrak{g}$ , which are independent of  $z$ ,  $\mathfrak{h}$ -invariant, and satisfy the generalized unitarity condition:  $r+r^{21}$  is a constant, invariant element of  $S^2\mathfrak{g}$ . To give geometric meaning to such solutions, we define and study the notion of a *dynamical Poisson groupoid* which is a special case of the notion of a Poisson groupoid introduced by Weinstein [W]. We show that any solution of (5) of the above type naturally defines a dynamical Poisson groupoid, which, as a manifold, equals  $U \times G \times U$ , where  $U \subset \mathfrak{h}^*$  is an open set. This construction illustrates how equation (5) arises naturally in the theory of Poisson groupoids. When  $\mathfrak{h} = 0$ , this construction reduces to the Drinfeld construction of a Poisson-Lie group from a solution of CYBE.

As in the case of the usual CYBE,  $z$ -dependent solutions of (5) define analogous structures on the loop group  $LG$ .

**0.4.** The second, more technically challenging goal, is (partially) attained in Chapters 3 and 4.

In Chapter 3, we consider  $z$ -independent solutions of CDYBE which satisfy the condition that  $r + r^{21}$  is a constant invariant element, when  $\mathfrak{g}$  is a simple finite-dimensional Lie algebra, or, more generally, a Kac-Moody algebra. In this case,  $r + r^{21} = \epsilon\Omega$ , where  $\Omega$  is the Casimir element, and  $\epsilon$  is a number called the coupling constant. For a simple Lie algebra  $\mathfrak{g}$ , we classify all solutions. It turns out that there are two types of solutions – rational (with zero coupling constant) and trigonometric (with nonzero coupling constant).

If  $\mathfrak{g}$  is an arbitrary Kac-Moody algebra, we classify solutions satisfying some additional conditions. Again, we find two types of solutions – rational and trigonometric.

We also classify invariant solutions of CDYBE for a pair of Lie algebras  $\mathfrak{l} \subset \mathfrak{g}$ , where  $\mathfrak{g}$  is a finite-dimensional simple Lie algebra, and  $\mathfrak{l}$  is a reductive Lie subalgebra of  $\mathfrak{g}$  containing the Cartan subalgebra  $\mathfrak{h}$ . The classification is obtained by reduction of CDYBE for the pair  $\mathfrak{l} \subset \mathfrak{g}$  to CDYBE for the pair  $\mathfrak{h} \subset \mathfrak{g}$ .

In Chapter 4 we are concerned with  $z$ -dependent solutions of CDYBE. We consider such solutions, satisfying the unitarity condition and the condition that the residue of  $r(\lambda, z)$  at  $z = 0$  equals  $\epsilon\Omega$ . As before,  $\epsilon$  is called the coupling constant. We classify all solutions with nonzero coupling constant. It turns out that there are three types of solutions – rational, trigonometric, and elliptic.

We also explain that  $z$ -independent solutions for the affine Lie algebra  $\tilde{\mathfrak{g}}$  can be interpreted as  $z$ -dependent solutions of CDYBE for  $\mathfrak{g}$  with  $\epsilon \neq 0$ .

**0.5.** The CDYB equation has a quantum analogue. This quantum equation is called the quantum dynamical Yang-Baxter equation (QDYBE). It was first introduced by Gervais and Neveu [GN] and later by Felder [F1], as a quantization of (5). This equation has important applications in the theory of integrable systems [ABB].

QDYBE is a generalization of the usual quantum Yang-Baxter equation, and it gives rise to the notion of a dynamical Hopf algebroid (or dynamical quantum groupoid), in the same way as the usual quantum Yang-Baxter equation gives rise to the notion of a Hopf algebra (quantum group). The notion of a dynamical Hopf algebroid is a quantization of the notion of a dynamical Poisson groupoid, discussed in this paper. An example of a dynamical Hopf algebroid, which is not a Hopf algebra, is the elliptic quantum group of [F1, FV].

In the following papers we will define and study the notion of a dynamical Hopf

algebroid, give examples, and show how such objects arise naturally in representation theory of affine Lie algebras and quantum groups, and in conformal field theory.

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## 1. DYNAMICAL POISSON GROUPOIDS.

### 1.1. Definition of a dynamical Poisson groupoid.

Recall that a groupoid is a category where all morphisms are isomorphisms. In this paper, we consider only groupoids whose objects and morphisms form a set, and not only a class (i.e. groupoids built on small categories).

Thus, a groupoid is defined by the following data: a set  $X$  (of morphisms, or arrows) called the groupoid itself, a set  $P$  (of objects) called the base of  $X$ , two surjective maps  $s, t : X \rightarrow P$  (source and target), a map  $m : \{(a, b) \in X \times X : s(a) = t(b)\} \rightarrow X$  (called the arrow composition map), and an injective map  $E : P \rightarrow X$  (the identity morphism;  $E(p) = id_p$ ), satisfying a number of obvious conditions. One of these conditions is the existence of an involution  $i : X \rightarrow X$  defined by the conditions  $s(i(x)) = t(x)$ ,  $s(x) = t(i(x))$ ,  $m(x, i(x)) = id_{t(x)}$ ,  $m(i(x), x) = id_{s(x)}$ . For brevity, when we talk about a particular groupoid, we will refer to its set of morphisms  $X$ .

A groupoid with one object is a group. Thus, the notion of a groupoid is a generalization of the notion of a group.

The role of the unit in a groupoid is played by the map  $E$ , and the role of inversion by  $i$ .

A Lie groupoid is a groupoid with a smooth structure (the sets of objects and morphisms are smooth manifolds, the structure maps are smooth, and some additional conditions, see [M]).

According to [W], a Poisson groupoid is a Lie groupoid  $X$  with a Poisson bracket such that the graph of the composition map (defined only on a subset of  $X \times X$ ) is a coisotropic submanifold of  $X \times X \times \bar{X}$ , where  $\bar{X}$  is the opposite Poisson manifold to  $X$ . For example, if  $|P| = 1$  (i.e.  $X$  is a Lie group), the structure of a Poisson groupoid is the same as a structure of a Poisson-Lie group.

In this section we will define a class of Poisson groupoids which we call dynamical Poisson groupoids.

Let  $G$  be a Lie group, and  $\mathfrak{g}$  its Lie algebra. Let  $H$  be a connected Lie subgroup of  $G$ , and  $\mathfrak{h}$  the Lie algebra of  $H$ . Let  $U$  be an open subset of  $\mathfrak{h}^*$ , which is invariant under the coadjoint action.

Consider the manifold  $X(G, H, U) = U \times G \times U$ . This manifold has a natural structure of a Lie groupoid:  $X = X(G, H, U)$ ,  $P = U$ ,  $s(u_1, g, u_2) = u_2$ ,  $t(u_1, g, u_2) = u_1$ ,  $E(u) = (u, 1, u)$ , and  $m((u_1, f, u_2), (u_2, g, u_3)) = (u_1, fg, u_3)$ . In the theory of groupoids, this groupoid is called the direct product of the trivial groupoid with base  $U$  and the group  $G$ .

The inversion of the groupoid  $X$  is given by  $i(u_1, g, u_2) = (u_2, g^{-1}, u_1)$ .

The manifold  $X$  carries a pair of commuting actions of  $H$ : the left action given by  $l(h)(u_1, g, u_2) = (hu_1h^{-1}, hg, u_2)$ , and the right action given by  $r(h)(u_1, g, u_2) =$

$(u_1, gh, h^{-1}u_2h)$  ( $huh^{-1}$  denotes, here and below, the coadjoint action of  $h$  on  $u$ ). The manifold  $X \times X$  carries a (left) action of  $H$  given by  $\Delta(h)(x, y) = (r(h)^{-1}x, l(h)y)$ . We will call this action the diagonal action. This action preserves the composition map.

For  $a \in \mathfrak{h}$ , let  $a_1, a_2$  be the functions on  $X$  given by  $a_1(u_1, g, u_2) = a(u_1)$ ,  $a_2(u_1, g, u_2) = a(u_2)$ .

Let  $\{, \}$  be a Poisson bracket on  $X$ .

**Definition.** The pair  $(X, \{, \})$  is said to be a dynamical Poisson groupoid if the following two conditions are satisfied.

(i) The actions  $l, r$  are Hamiltonian, the maps  $t, s$  are moment maps for them, and for any  $a, b \in \mathfrak{h}$  one has  $\{a_1, b_2\} = 0$ .

(ii) Let  $X \bullet X := X \times X / \Delta(H)$  be the Hamiltonian reduction of  $X \times X$  by the diagonal action of  $H$ , and let  $\bar{m} : X \bullet X \rightarrow X$  be the reduction by  $H$  of the composition map  $m$  of  $X$ . Then  $\bar{m}$  is a Poisson map.

**Remark 2.** Let us explain condition (ii). If condition (i) is satisfied, the diagonal action of  $H$  is Hamiltonian, and  $\mu(x, y) = t(y) - s(x)$  is a moment map for this action. Therefore, the set of zeros of the moment map,  $\mu^{-1}(0)$ , is precisely the domain of the map  $m$ . Thus, the definition of  $\bar{m}$  makes sense. The space  $X \bullet X$  equals  $U \times (Y/H) \times U$ , where  $Y = G \times U \times G$ , and  $H$  acts on  $Y$  by  $h \circ (f, u, g) = (fh^{-1}, huh^{-1}, hg)$ . The space  $X \bullet X$  has the Poisson structure of the Hamiltonian reduction. Therefore, it makes sense to require that the map  $\bar{m}$  is Poisson.

If  $H = 1$ , then  $\{, \}$  is a Poisson bracket on  $G$ . Condition (i) is vacuous, and condition (ii) says that the multiplication map in  $G$  is Poisson. Thus, a dynamical Poisson groupoid with  $H = 1$  is simply a Poisson-Lie group.

Let us compute the general form of the Poisson bracket on a dynamical Poisson groupoid  $X = U \times G \times U$ . Let  $f$  be any function on  $X$  which is pulled back from  $G$ . By the definition, we have the following Poisson commutation relations:

$$(1.1) \quad \{a_1, b_2\} = 0, \{a_1, b_1\} = -[a, b]_1, \{a_2, b_2\} = [a, b]_2, \{a_1, f\} = R_a f, \{a_2, f\} = L_a f,$$

where  $L_a, R_a$  are the left- and the right-invariant vector fields on  $G$  corresponding to  $a$ . So the only freedom that we have is in the bracket of functions on  $G$ .

## 1.2. The Hamiltonian unit.

For Poisson Lie groups, it is known that the unit  $E : \{1\} \rightarrow G$  is a Poisson map. In the case of dynamical Poisson groupoids, this property fails: the image of  $E$  is not a Poisson submanifold of  $X$ . Therefore it makes sense to extend  $E$  so that its image becomes the smallest Poisson manifold containing the image of  $E$ . This is done by introducing the Hamiltonian unit.

Let  $(T^*H)_U$  be the set of all  $(h, p) \in T^*H$  ( $h \in H, p \in T_h^*H$ ) such that  $h^{-1}p \in U$ . We equip  $(T^*H)_U$  with the standard symplectic structure.

**Definition.** The Hamiltonian unit of a dynamical Poisson groupoid  $X$  is the map  $e : (T^*H)_U \rightarrow X$  given by  $e(h, p) = (ph^{-1}, h, h^{-1}p)$ .

It is easy to check that this map is Poisson, and its image is the smallest Poisson submanifold of  $X$  containing the image of  $E$ .

## 1.3. Poisson groupoids and the CDYB equation.

In this section we will construct examples of dynamical Poisson groupoids, and will be naturally led to the classical dynamical Yang-Baxter equation (CDYBE).

We work in the setting of Section 1.1. Namely, we are considering the Lie groupoid  $X = X(G, H, U)$ . We want to make  $X$  into a dynamical Poisson groupoid. As we know, the Poisson bracket on  $X$  is partially defined by (1.1), and it remains to define the Poisson bracket of two functions pulled back from  $G$ .

Recall [Dr] that if  $K$  is a Poisson Lie group, and  $\mathfrak{k}$  its Lie algebra, then a coboundary Poisson-Lie structure on  $K$  is a Poisson-Lie structure with the Poisson bivector of the form

$$(1.2) \quad \pi = R(\rho) - L(\rho),$$

where  $\rho \in \Lambda^2 \mathfrak{k}$ , and  $L(\rho), R(\rho)$  are the left- and the right-invariant tensor fields equal to  $\rho$  at the identity. It is known [Dr] that (1.2) defines a Poisson-Lie structure iff

$$(1.3) \quad CYB(\rho) := [\rho^{12}, \rho^{13}] + [\rho^{12}, \rho^{23}] + [\rho^{13}, \rho^{23}] \in (\Lambda^3 \mathfrak{k})^{\mathfrak{k}}.$$

By analogy with this definition, we will look for a Poisson bracket  $\pi$  on  $X$  such that for any functions  $f_1, f_2$  pulled back from  $G$

$$(1.4) \quad \{f_1, f_2\} = (df_1 \otimes df_2)(R(\rho(u_1)) - L(\rho(u_2))),$$

where  $\rho : U \rightarrow \Lambda^2 \mathfrak{g}$  is a smooth function.

For a Lie algebra  $\mathfrak{g}$  and a tensor  $Z \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ , define

$$\text{Alt}(Z) = Z^{123} + Z^{231} + Z^{312}.$$

Let  $\rho : U \rightarrow \Lambda^2 \mathfrak{g}$  be a smooth function. Then the differential  $d\rho$  is a 1-form on  $U$  with coefficients in  $\Lambda^2 \mathfrak{g}$ , so it can be considered as a function on  $U$  with values in  $\mathfrak{h} \otimes \Lambda^2 \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ .

Define the classical dynamical Yang-Baxter functional

$$(1.5) \quad CDYB(\rho) = \text{Alt}(d\rho) + CYB(\rho).$$

**Theorem 1.1.** *Formulas (1.1), (1.4) define a Poisson structure on  $X$  if and only if*

- (i)  $\rho$  is  $H$ -equivariant, and
- (ii) the element  $Z = CDYB(\rho(u))$  is constant (in  $u$ ) and lies in  $(\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$ .

*Proof.* Property (i) is equivalent to the Jacobi identity for three functions  $a_1, f_1, f_2$ , where  $a \in \mathfrak{h}$ , and  $f_1, f_2$  are pulled back from  $G$  (here, as usually,  $a$  is regarded as a linear function on  $U$ ). Also, when (i) is satisfied, then (ii) is equivalent to the Jacobi identity for three functions  $f_1, f_2, f_3$  pulled back from  $G$ .

Now let  $\rho$  satisfy conditions (i), (ii). Then  $X$  equipped with the Poisson bracket  $\{\cdot, \cdot\}$  defined by  $\rho$  is a dynamical Poisson groupoid. Indeed, it is easy to see that the composition map  $m : X \bullet X \rightarrow X$  given by  $(u_1, g_1, u_2) \bullet (u_2, g_2, u_3) = (u_1, g_1 g_2, u_3)$  is Poisson.

**Definition.** A dynamical Poisson groupoid defined by (1.4) is called a coboundary dynamical Poisson groupoid.

Recall [Dr] that a coboundary Poisson-Lie group  $K$  defined by (1.2) is called quasitriangular if it is equipped with a constant element  $T \in (S^2\mathfrak{g})^{\mathfrak{g}}$ , such that  $CYB(\rho) = \frac{1}{4}[T^{12}, T^{23}]$ . In this case the element  $r = \rho + \frac{1}{2}T$  is a solution of the classical Yang-Baxter equation, and is called the classical r-matrix of  $G$ . If  $T = 0$ , then the quasitriangular structure defined by  $T$  is called triangular.

Thus, quasitriangular structures on  $G$  are parametrized by solutions  $r$  of the CYB equation, such that  $r + r^{21}$  is a  $\mathfrak{g}$ -invariant element in  $S^2\mathfrak{g}$ . Triangular structures are parametrized by skew-symmetric solutions of the CYB equation.

**Definition.** A coboundary dynamical Poisson groupoid is called quasitriangular if it is equipped with a constant element  $T \in (S^2\mathfrak{g})^{\mathfrak{g}}$ , such that  $CDYB(\rho) = \frac{1}{4}[T^{12}, T^{23}]$ . If  $T = 0$ , the quasitriangular structure defined by  $T$  is called triangular.

In the quasitriangular case the function  $r = \rho + \frac{1}{2}T$  is a solution of the classical dynamical Yang-Baxter equation

$$(1.6) \quad CDYB(r) = 0.$$

and is called the classical dynamical r-matrix of  $X$ . Thus, quasitriangular structures on  $X$  are parametrized by solutions  $r$  of the CDYB equation, which are  $\mathfrak{h}$ -invariant and such that  $r + r^{21}$  is a constant  $\mathfrak{g}$ -invariant element in  $S^2\mathfrak{g}$ . Triangular structures are parametrized by skew-symmetric,  $\mathfrak{h}$ -invariant solutions of the CDYB equation.

**Remark.** The material of this section trivially generalizes to the case when  $G, H$  are complex Lie groups (algebraic groups, formal groups) rather than real Lie groups.

#### 1.4. Gauge transformations of coboundary dynamical Poisson groupoids.

Let  $G$  be a complex Lie group,  $H$  a commutative, connected complex Lie subgroup of  $G$ ,  $\mathfrak{g}, \mathfrak{h}$  their Lie algebras, and  $U \subset \mathfrak{h}^*$  a connected open set. Let  $G^H$  be the centralizer of  $H$  in  $G$ , and  $\mathfrak{g}^H$  its Lie algebra.

In this section we will use the following notation: If  $\alpha$  is a  $k$ -form on  $U$  with values in a vector space  $W$ , then  $\bar{\alpha}$  is the corresponding function  $U \rightarrow \Lambda^k\mathfrak{h} \otimes W$ .

Let  $CB(G, H, U)$  be the set of all coboundary dynamical Poisson structures on the groupoid  $X = X(G, H, U)$ . That is,  $CB(G, H, U)$  is the set of all  $\mathfrak{h}$ -invariant holomorphic functions  $\rho : U \rightarrow \Lambda^2\mathfrak{g}$  such that  $CDYB(\rho) = Z \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$  is a constant.

It turns out that there is a natural (infinite-dimensional) group which acts on  $CB(G, H, U)$ , such that the space of its orbits is finite-dimensional.

Let  $g : U \rightarrow G^H$  be a holomorphic function. Let  $\eta = g^{-1}dg$  be a 1-form on  $U$  with values in  $\mathfrak{g}^H$ . The form  $\eta$  defines a function  $\bar{\eta} : U \rightarrow \mathfrak{h} \otimes \mathfrak{g}^H$ . For any function  $\rho : U \rightarrow \Lambda^2\mathfrak{g}$ , define

$$(1.7) \quad \rho^g := (g \otimes g)(\rho - \bar{\eta} + \bar{\eta}^{21})(g^{-1} \otimes g^{-1}).$$

**Proposition 1.2.** *If  $\rho \in CB(G, H, U)$  then  $\rho^g \in CB(G, H, U)$ , and  $CDYB(\rho^g) = CDYB(\rho)$ .*

*Proof.* Fix a basis  $\{x_i\}$  of  $\mathfrak{h}$ . We have

$$(1.8) \quad \begin{aligned} CDYB(\rho^g) &= (Adg)^{\otimes 3} \left( CDYB(\rho) + CDYB(\bar{\eta}^{21} - \bar{\eta}) \right. \\ &\quad - [\rho^{12}, \bar{\eta}^{23} - \bar{\eta}^{32}] - [\rho^{12}, \bar{\eta}^{13} - \bar{\eta}^{31}] - [\rho^{13}, \bar{\eta}^{23} - \bar{\eta}^{32}] + \\ &\quad \left. [\rho^{23}, \bar{\eta}^{12} - \bar{\eta}^{21}] + [\rho^{13}, \bar{\eta}^{12} - \bar{\eta}^{21}] + [\rho^{23}, \bar{\eta}^{13} - \bar{\eta}^{31}] + \right. \\ &\quad \left. Alt \left( \sum x_i \otimes [g^{-1} \frac{\partial g}{\partial x_i} \otimes 1 + 1 \otimes g^{-1} \frac{\partial g}{\partial x_i}, \rho + \bar{\eta} - \bar{\eta}^{21}] \right) \right). \end{aligned}$$

Using the facts that  $\rho$  is invariant under  $\mathfrak{h}$ , and  $CDYB(\rho)$  is invariant under  $G$ , we have

$$(1.9) \quad \begin{aligned} (Adg^{-1})^{\otimes 3} (CDYB(\rho^g) - CDYB(\rho)) &= CDYB(\bar{\eta}^{21} - \bar{\eta}) + Alt([\rho^{23}, \bar{\eta}^{12} + \bar{\eta}^{13}]) + \\ &Alt \left( \sum x_i \otimes [g^{-1} \frac{\partial g}{\partial x_i} \otimes 1 + 1 \otimes g^{-1} \frac{\partial g}{\partial x_i}, \rho - \bar{\eta} + \bar{\eta}^{21}] \right). \end{aligned}$$

Simplifying the last two terms in (1.9), we get

$$(1.10) \quad \begin{aligned} (Adg^{-1})^{\otimes 3} (CDYB(\rho^g) - CDYB(\rho)) &= \\ &CDYB(\bar{\eta}^{21} - \bar{\eta}) + Alt([\bar{\eta}^{12} + \bar{\eta}^{13}, \bar{\eta}^{32} - \bar{\eta}^{23}]). \end{aligned}$$

However, since  $\bar{\eta} \in \mathfrak{h} \otimes \mathfrak{g}^H$ , we have  $[\bar{\eta}^{12}, \bar{\eta}^{13}] = [\bar{\eta}^{12}, \bar{\eta}^{23}] = 0$ . Therefore, we have

$$(1.11) \quad \begin{aligned} (Adg^{-1})^{\otimes 3} (CDYB(\rho^g) - CDYB(\rho)) &= \\ &CDYB(\bar{\eta}^{21} - \bar{\eta}) + Alt([\bar{\eta}^{12}, \bar{\eta}^{32}] - [\bar{\eta}^{13}, \bar{\eta}^{23}]) = Alt(d\bar{\eta}^{21} - d\bar{\eta} - [\bar{\eta}^{13}, \bar{\eta}^{23}]). \end{aligned}$$

Let  $\bar{F}_\eta : U \rightarrow \mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{g}^H$  be the function corresponding to the curvature form  $F_\eta = d\eta + \frac{1}{2}[\eta, \eta]$ . It is easy to see that the r.h.s. of (1.11) equals  $-Alt(\bar{F}_\eta)$ . Finally, observe that by the definition the form  $\eta$  satisfies the zero-curvature condition  $F_\eta = 0$ . Thus,  $CDYB(\rho^g) = CDYB(\rho)$ .  $\square$

It is easy to check that  $(\rho^f)^g = \rho^{gf}$  for  $f, g : U \rightarrow G^H$ , so the assignment  $\rho \rightarrow \rho^g$  defines a left action of the group  $\Sigma := Map(U, G^H)$  of holomorphic functions on  $U$  with values in  $G^H$  on  $CB(G, H, U)$ .

**Definition.** *We will call transformations  $\rho \rightarrow \rho^g$  the gauge transformations of the first kind.*

**Remark.** The gauge transformations of the first kind are especially simple if  $g$  takes values in  $H \subset G^H$ . In this case  $\rho^g = \rho + \bar{\eta}^{21} - \bar{\eta}$ .

Now let  $\omega$  be a closed holomorphic 2-form on  $U$ . This form defines a holomorphic function  $\bar{\omega} : U \rightarrow \Lambda^2 \mathfrak{g}$ . To this function there corresponds a transformation of  $CB(G, H, U)$ , given by  $\rho \rightarrow \rho + \bar{\omega}$ . We will call such transformations gauge transformations of the second kind.



**Proposition 1.3.** *If the form  $\omega$  is exact, then the gauge transformation of the second kind  $\rho \rightarrow \rho + \bar{\omega}$  is also of the first kind.*

*Proof.* Let  $\xi$  be a 1-form on  $U$  such that  $\omega = d\xi$ . This 1-form defines a function  $\bar{\xi} : U \rightarrow \mathfrak{h}$ . Define a holomorphic function  $g_\xi : U \rightarrow H$  by  $g_\xi = e^{-\bar{\xi}}$ . Then  $\eta = g_\xi^{-1}dg_\xi = -d\bar{\xi}$ . Thus,  $\bar{\eta}^{21} - \bar{\eta} = \overline{d\xi} = \bar{\omega}$ , as desired.  $\square$

From now till the end of this section we will assume that  $U$  is the formal polydisc. In this case by holomorphic functions we will mean arbitrary formal power series. Therefore, the constructions below make sense not only for the field  $\mathbb{C}$  but for any field of characteristic zero.

Since  $U$  is a formal polydisc, any closed form is exact. Thus, gauge transformations of the second kind are also of the first kind, and we will call them simply gauge transformations.

Now we will show that the quotient space  $CB(G, H, U)/\Sigma$  is finite-dimensional.

**Theorem 1.4.** *Let  $\rho, r \in CB(G, H, U)$ . Assume that the values of  $\rho, r$  at the origin are equal, and  $CDYB(\rho) = CDYB(r)$ . Then  $\rho = r^g$  for some  $g \in \Sigma$ .*

*Proof.* Let  $x_1, \dots, x_n$  be a basis of  $\mathfrak{h}$ . We regard it as a system of linear coordinates on  $U$ . The functions  $r, \rho$  are by definition formal power series in  $x_i$ .

We will prove, by induction in  $N$ , that the statement of the theorem holds modulo terms of order  $N + 1$  of the power series. This is enough to prove the theorem.

For  $N = 0$ , the statement follows from the assumption of the theorem. Suppose we know it for  $N = K$ , and let us prove it for  $N = K + 1$ .

We know that there exists a gauge transformation  $g_K \in \Sigma$  such that the error term  $\epsilon_K := \rho - r^{g_K}$  is of order  $K + 1$ . Let  $\epsilon_K = E_K + \tilde{\epsilon}_K$ , where  $E_K$  is the part of  $\epsilon_K$  of degree exactly  $K + 1$ .

Since  $\rho$  and  $r_K := r^{g_K}$  satisfy the property  $CDYB(\rho) = CDYB(r_K)$ , we have

$$Alt(dE_K) = [CYB(r_K) - CYB(\rho)]_K,$$

where  $[*]_K$  denotes the degree  $K$  homogeneous part of an expression  $*$ . But according to our assumption,  $\rho$  and  $r_K$  coincide in degrees  $\leq K$ , so we get

$$Alt(dE_K) = 0.$$

Now we will study the equation  $Alt(dE) = 0$ , which is the linearization of the CDYB equation near the zero solution.

**Lemma 1.** Let  $E : U \rightarrow \Lambda^2 \mathfrak{g}$  be a homogeneous polynomial function of degree  $\geq 1$ , invariant under  $\mathfrak{h}$ , such that  $Alt(dE) = 0$ . Then  $E$  takes values in  $\mathfrak{h} \otimes \mathfrak{g}^H + \mathfrak{g}^H \otimes \mathfrak{h}$ .

*Proof.* Because of the  $\mathfrak{h}$ -invariance of  $E$ , it is enough to show that  $E \in \mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}$ .

Let  $V \subset \mathfrak{g}$  be a vector subspace which is a complement to  $\mathfrak{h}$ . Since  $E(u) \in \Lambda^2 \mathfrak{g}$ , we can write  $E$  uniquely as a sum  $E = E_{VV} + E_{V\mathfrak{h}} + E_{\mathfrak{h}V} + E_{\mathfrak{h}\mathfrak{h}}$ , where  $E_{VV} \in \Lambda^2 V$ ,  $E_{\mathfrak{h}\mathfrak{h}} \in \Lambda^2 \mathfrak{h}$ ,  $E_{V\mathfrak{h}} \in V \otimes \mathfrak{h}$ , and  $E_{\mathfrak{h}V} = -E_{V\mathfrak{h}}^{21}$ .

The equation  $Alt(dE) = 0$  splits in 3 parts: the  $\mathfrak{h}VV$ -part, the  $\mathfrak{h}\mathfrak{h}V$ -part, and the  $\mathfrak{h}\mathfrak{h}\mathfrak{h}$ -part.

The  $\mathfrak{h}VV$ -part says that  $dE_{VV} = 0$ , i.e.  $E_{VV}$  is a constant. Since  $E_{VV}$  is homogeneous of degree  $\geq 1$ ,  $E_{VV} = 0$ . The Lemma is proved.

**Lemma 2.** In the situation of Lemma 1, there exists a closed 1-form  $\eta$  on  $U$  with values in  $\mathfrak{g}^H$ , homogeneous of degree  $K$ , such that  $E = \bar{\eta}^{21} - \bar{\eta}$ .

*Proof.* Let  $V^H = V \cap \mathfrak{g}^H$ . Then  $V^H$  is a complement to  $\mathfrak{h}$  in  $V$ . Let  $\xi$  be the 1-form on  $U$  with values in  $V^H$  such that  $\bar{\xi} = -E|_V$ . The  $\mathfrak{h}\mathfrak{h}V$ -part of the equation  $\text{Alt}(dE) = 0$  says that  $d\xi = 0$ . Thus,  $\xi$  satisfies the conditions of Lemma 2. Therefore, it is enough to prove Lemma 2 for  $E' = E + \xi^{21} - \bar{\xi}$ .

By the definition,  $E'$  takes values in  $\Lambda^2\mathfrak{h}$ , and  $\text{Alt}(dE') = 0$ . Therefore,  $E'$  defines a closed 2-form on  $U$ , which we denote by  $\omega$  (we have  $\bar{\omega} = E'$ ). Let  $\zeta$  be a 1-form such that  $\omega = d\zeta$ , and let  $\theta = d\bar{\zeta}$ . Then  $\bar{\theta} - \theta^{21} = \bar{\omega} = E'$ . The Lemma is proved.

**Lemma 3.** Let  $\eta$  be a closed 1-form on  $U$  with coefficients in a complex finite dimensional Lie algebra  $\mathfrak{b}$ , which has order  $K$  at 0. Then there exists a 1-form  $\tau$  satisfying the zero-curvature equation  $d\tau + \frac{1}{2}[\tau, \tau] = 0$  such that  $\tau - \eta$  is of order  $\geq K + 1$ .

*Proof.* Choose a function  $\chi : U \rightarrow \mathfrak{b}$ , of order  $K + 1$ , such that  $\eta = d\chi$ . Let  $B$  be the Lie group of  $\mathfrak{b}$ , and consider the function  $g = e^\chi : U \rightarrow B$ . Set  $\tau = g^{-1}dg$ . Then  $\tau$  is the desired form.

Now we return to the proof of the theorem. We start with the function  $E = E_K$ . Let  $\eta$  be as in Lemma 2, and  $\tau$  as in Lemma 3. Let  $g : U \rightarrow G^H$  be the function such that  $g^{-1}dg = \tau$ . It is easy to see that for any  $s : U \rightarrow \Lambda^2\mathfrak{g}$  we have  $s^g - s = E + \epsilon$ , where  $\epsilon$  are terms of order  $K + 1$  and higher. In particular,  $r^{gg^K} - r^{g^K}$  equals  $E$  modulo terms of order  $\geq K + 1$ . Thus, if we set  $g_{K+1} = gg^K$ , we will get that  $\rho - r^{g_{K+1}}$  is of order  $\geq K + 1$ .

The theorem is proved.  $\square$

## 2. BIEQUIVARIANT POISSON MANIFOLDS AND GROUPOIDS.

In this Chapter we will introduce the notion of an  $H$ -biequivariant Poisson groupoid, which is a natural generalization of the notion of a dynamical Poisson groupoid.

### 2.1. Biequivariant Poisson manifolds.

**Definition.** An  $H$ -biequivariant Poisson manifold is a Poisson manifold  $X$  equipped with two commuting, proper, free Hamiltonian actions of  $H$  – a left action  $l : H \times X \rightarrow X$  and a right action  $r : X \times H \rightarrow X$ , and two maps  $\mu_l, \mu_r : X \rightarrow \mathfrak{h}^*$ , which are moment maps for  $l, r$ , such that for any smooth functions  $a, b$  on  $\mathfrak{h}^*$  one has  $\{a \circ \mu_l, b \circ \mu_r\} = 0$ .

**Remark.** We remind ([GHV], v.2, p. 135) that a smooth action of a group  $H$  on a manifold  $X$  is called proper if for any two compact sets  $A, B \in X$  the set of elements of  $a \in H$  such that  $B \cap aA \neq \emptyset$  is compact. It is known that if an action is proper and free then the quotient  $X/H$  has a unique structure of a smooth manifold such that the natural map  $X \rightarrow X/H$  is a smooth submersion.

We will denote the left and the right action of  $h \in H$  on  $x \in X$  by  $hx$  and  $xh$ , respectively.

Observe that for any  $H$ -biequivariant Poisson manifold  $X$  the maps  $\mu_l, \mu_r$  are submersions, since the actions  $l, r$  are free.

Let  $U \subset \mathfrak{h}^*$  be an open set invariant under the coadjoint action. We will say that an  $H$ -biequivariant Poisson manifold  $X$  is over  $U$  if the images of  $\mu_l, \mu_r$  coincide with  $U$ .

Let  $\mathcal{C}_U$  be the category of  $H$ -biequivariant Poisson manifolds over  $U$ , where morphisms are Poisson maps which commute with  $l, r, \mu_l, \mu_r$ . We will now define the structure of a monoidal category on  $\mathcal{C}_U$ .

Define the product  $\bullet$  on  $\mathcal{C}_U$  as follows. Let  $X_1, X_2 \in \mathcal{C}_U$ , with actions of  $H$   $l_1, r_1, l_2, r_2$ , and moment maps  $\mu_l^1, \mu_r^1, \mu_l^2, \mu_r^2$ . Consider the left action  $\Delta$  of  $H$  on  $X_1 \times X_2$  by  $\Delta(h)(x_1, x_2) = (x_1 h^{-1}, h x_2)$ . The moment map for this action is  $\mu_r^1 - \mu_l^2$ . Let  $X_1 \bullet X_2 = X_1 \times X_2 // H$  be the Hamiltonian reduction of  $X_1 \times X_2$  with respect to the action  $\Delta$  of  $H$ . That is,  $X_1 \bullet X_2 = Z/H$ , where  $Z \subset X_1 \times X_2$  is the set of points  $(x_1, x_2)$  such that  $\mu_r(x_1) = \mu_l(x_2)$ .

It is easy to see that  $Z$  is a smooth manifold (as  $\mu_l, \mu_r$  are submersions), and  $H$  acts freely and properly on  $Z$ , so  $Z/H$  is smooth.

The space  $X_1 \bullet X_2$  has a natural structure of an object of  $\mathcal{C}_U$ : it has a Poisson structure of Hamiltonian reduction, two free commuting actions of  $H$  –  $l_1$  and  $r_2$ , and two moment maps  $\mu_l^1$  and  $\mu_r^2$  for them. Thus,  $\bullet$  is a bifunctor  $\mathcal{C}_U \times \mathcal{C}_U \rightarrow \mathcal{C}_U$ .

It is easy to see that the operation  $\bullet$  is associative:  $X_1 \bullet (X_2 \bullet X_3) = (X_1 \bullet X_2) \bullet X_3 = X_1 \times X_2 \times X_3 // H \times H$ , where  $H \times H$  acts on  $X_1 \times X_2 \times X_3$  by  $(h_1, h_2)(x_1, x_2, x_3) = (x_1 h_1^{-1}, h_1 x_2 h_2^{-1}, h_2 x_3)$ .

Define an object  $\mathbf{1} \in \mathcal{C}_U$  to be  $(T^*H)_U$  (see Chapter 1), with the obvious left and right actions of  $H$ , and the moment maps  $\mu_l(p, h) = ph^{-1}$ ,  $\mu_r(p, h) = h^{-1}p$ ,  $h \in H, p \in T_h^*H$ . It turns out that  $\mathbf{1}$  is a unit object of  $\mathcal{C}_U$ .

Indeed, let us check that  $\mathbf{1} \bullet X$  is naturally isomorphic to  $X$ . We have  $Z = \{(h, p, x) : h^{-1}p = \mu_l(x)\}$ . Thus,  $Z$  is naturally identified with  $H \times X$ , via  $(h, p, x) \rightarrow (h, x)$ . The action of  $H$  on  $Z$  is by  $h(h', x) = (h'h^{-1}, hx)$ . Thus, the quotient  $Z/H$  is naturally isomorphic to  $X$ , and it is easy to check that the Poisson structure on  $X$ , the two actions of  $H$ , and the corresponding moment maps are the original ones. Similarly one checks that  $X \bullet \mathbf{1}$  is naturally isomorphic to  $X$ . The unit object axioms are checked directly. Thus,  $(\mathcal{C}_U, \bullet, \mathbf{1})$  is a monoidal category with a unit object  $\mathbf{1}$ .

**Remark.** Let  $\mathcal{D}$  be the category whose objects are manifolds with two commuting, free, proper actions of  $H$  – a left action  $l$  and a right action  $r$ . This category has a natural monoidal structure, with product being the fiber product  $\times_H$  over  $H$ , and the unit object  $H$  (with obvious  $l$  and  $r$ ). Then we have a natural functor  $T^*$  from  $\mathcal{D}$  to  $\mathcal{C}_{\mathfrak{h}^*}$  – the functor of cotangent bundle:  $M \rightarrow T^*M$ . (Indeed, it is well known that if  $H$  acts on  $M$ , then its induced action on  $T^*M$  is Hamiltonian, with moment map  $\mu(m, p)(L) = \langle L(m), p \rangle$ , where  $m \in M, p \in T_m^*M, L \in \mathfrak{h}$ , and  $L(a), a \in M$  is the corresponding vector field on  $M$ . Thus,  $T^*M$  is naturally an object of  $\mathcal{C}_{\mathfrak{h}^*}$ .) The main property of the functor  $T^*$  is that it is a monoidal functor:  $T^*(M_1 \times_H M_2) = T^*M_1 \bullet T^*M_2$ .

Let  $X \in \mathcal{C}_U$ . Denote by  $\bar{X}$  the new object of  $\mathcal{C}_U$  obtained as follows:  $\bar{X}$  is  $X$  as a manifold, with the opposite Poisson structure  $-\{\cdot, \cdot\}$ , the left and the right actions of  $H$  permuted (i.e. the left, respectively right, action of  $h$  on  $\bar{X}$  is the right, respectively left, action of  $h^{-1}$  on  $X$ ), and the moment maps also permuted. We will call  $\bar{X}$  the dual object to  $X$ . By a reflection on  $X$  we will mean an involutive morphism  $i : X \rightarrow \bar{X}$ . We will often write  $x^{-1}$  for  $i(x)$ .

Let  $X \in \mathcal{C}_U$  and  $i : X \rightarrow \bar{X}$  be a reflection map. Let  $\varphi_+^i, \varphi_-^i : X \rightarrow X \times X$  be given by the formulas  $\varphi_+^i(x) = (x, x^{-1})$ ,  $\varphi_-^i(x) = (x^{-1}, x)$ . It is easy to see that these maps descend to maps (not necessarily Poisson)  $\psi_{\pm}^i : X \rightarrow X \bullet X$ , as

$$\mu_r(x^{-1}) = \mu_l(x), \mu_l(x^{-1}) = \mu_r(x).$$

## 2.2. Biequivariant Poisson groupoids.

**Definition.** Let  $X \in \mathcal{C}_U$ . We will say that  $X$  is an  $H$ -biequivariant Poisson semi-groupoid if it is equipped with an associative morphism  $m : X \bullet X \rightarrow X$  (called multiplication). In this case, a unit in  $X$  is a morphism  $e : \mathbf{1} \rightarrow X$  such that the morphisms  $m \circ (e \bullet \text{id})$ ,  $m \circ (\text{id} \bullet e)$  are the canonical morphisms  $\mathbf{1} \times X \rightarrow X$ ,  $X \times \mathbf{1} \rightarrow X$ . Further, such  $X$  is called an  $H$ -biequivariant Poisson groupoid if it is equipped with a reflection  $i : X \rightarrow \bar{X}$  (called the inversion map), such that  $m(\psi_+^i(x)) = e(1, \mu_l(x))$ ,  $m(\psi_-^i(x)) = e(1, \mu_r(x))$ , where  $(1, \mu_{l,r}(x)) \in (T_1^*H)_U$ .

**Remark 1.** If  $X$  has a unit, it is unique, as for any two units  $e_1, e_2$ , we have  $e_1 = m \circ (e_1 \bullet e_2) = e_2$ . If the inversion map  $i$  on  $X$  exists, it is also unique, as for any two inversion maps  $i_1, i_2$ ,  $m_3(i_1(x) \bullet x \bullet i_2(x)) = i_1(x) = i_2(x)$ , where  $m_3 : X \bullet X \bullet X \rightarrow X$  is the multiplication map.

**Remark 2.** If  $H$  is trivial, the notion of an  $H$ -biequivariant Poisson groupoid coincides with the notion of a Poisson-Lie group.

If  $X$  is a dynamical Poisson groupoid, it is automatically an  $H$ -biequivariant Poisson groupoid. Indeed, in this case  $X$  is an  $H$ -biequivariant Poisson manifold, with  $\mu_l = t, \mu_r = s$ . The composition map is the map  $\bar{m}$  defined in Section 1.1, which is obviously associative. The unit axiom is satisfied for the Hamiltonian unit  $e : \mathbf{1} \rightarrow X$ . Finally, it is easy to check that the inversion map is anti-Poisson and satisfies the inversion axiom.

As we mentioned, the notion of a Poisson groupoid is known in the literature [W]. So let us justify the usage of this term in our paper, by showing that  $X$  is indeed a Poisson groupoid in the sense of [W] in a natural way.

Let  $X$  be an  $H$ -biequivariant Poisson groupoid over  $U$ . Then  $X$  has a natural structure of groupoid in the usual sense. Namely,  $P = U$ ,  $s = \mu_r, t = \mu_l$ , the composition map is given by the multiplication map  $m$ , and  $E = e|_U$ , where  $U = \{(1, p) \in (T^*H)_U\}$ . It is easy to check that this groupoid is in fact a Lie groupoid in the sense of [M].

**Proposition 2.1.**  $X$  is a Poisson groupoid in the sense of [W].

*Proof.* We should show that the graph of composition is coisotropic. This follows from the following easy lemma.

**Lemma.** Let  $X$  be a Poisson manifold with a proper, free hamiltonian action of a connected Lie group  $H$  with moment map  $\mu : X \rightarrow \mathfrak{h}^*$ . Let  $X_0 = \mu^{-1}(0)$ . Let  $Y$  be another Poisson manifold, and  $f : X_0 \rightarrow Y$  be an  $H$ -invariant smooth map. Then:  $f$  descends to a Poisson map  $X//H \rightarrow Y$  (where  $X//H$  is the hamiltonian reduction) if and only if the graph of  $f$  is coisotropic in  $X \times \bar{Y}$  (where  $\bar{Y}$  is  $Y$  with the opposite Poisson structure).

*Proof.* Straightforward.

To prove Proposition 2.1, it is enough to apply this Lemma to  $f = m$ , where  $m$  is the multiplication map in the groupoid.

**Remark.** We have defined dynamical and  $H$ -biequivariant Poisson groupoids in the category of smooth manifolds. Similarly one can define the same objects in the categories of complex analytic, formal, and algebraic varieties. In the formal and algebraic settings, we can work over an arbitrary field of characteristic zero. These generalizations are straightforward, and we will not give them here.

### 2.3. $H$ -biequivariant Poisson algebras.

In this and the next section we will sketch the constructions of the previous two sections in the algebraic language, i.e. working with Poisson algebras rather than Poisson manifolds. This is related to the previous two sections by the operation of taking spectrum.

Let  $k$  be a field of characteristic zero. Let  $A$  be a Poisson algebra over  $k$ ,  $H$  a connected affine algebraic group, and  $\psi : H \times A \rightarrow A$  be a right algebraic action of  $H$  on  $A$  by Poisson automorphisms. (“Algebraic” means that  $A$  is a sum of finite dimensional representations of  $H$ ).

Let  $\mathfrak{h}$  be the Lie algebra of  $H$ . Then the variety  $\mathfrak{h}^*$  has a natural Poisson structure. Let  $U \subset \mathfrak{h}^*$  be an  $H$ -invariant open set. A Poisson homomorphism  $\mu : \mathcal{O}(U) \rightarrow A$  (where  $\mathcal{O}(X)$  denotes the ring of algebraic functions on a variety  $X$ ) is called a *moment map* for  $\psi$  if for any regular function  $g$  on  $U$ , and any  $f \in A$  we have

$$\{\mu(g), f\} = \sum_j \mu\left(\frac{\partial g}{\partial y_j}\right) \cdot d\psi|_{h=1}(y_j, f).$$

Here  $y_j \in \mathfrak{h}$  are a linear system of coordinates on  $U$ ,  $h \in H$ , and  $d\psi|_{h=1} : \mathfrak{h} \times A \rightarrow A$  is the differential of  $\psi$  at  $h = 1 \in H$ . In particular, for a linear function on  $U$  given by  $a \in \mathfrak{h}$  the last equation is

$$\{\mu(a), f\} = \sum_j d\psi|_{h=1}(a, f).$$

For a left action of  $H$ , a moment map is defined in the same way, with the only difference that it is anti-Poisson rather than Poisson.

**Definition.** An  $H$ -biequivariant Poisson algebra over  $U$  is a 5-tuple  $(A, l, r, \mu_l, \mu_r)$ , where  $A$  is a Poisson algebra with 1 over  $k$ ,  $l, r$  is a pair of commuting algebraic actions of  $H$  on  $A$  (a left action and a right action) by Poisson algebra automorphisms, and  $\mu_l, \mu_r : \mathcal{O}(U) \rightarrow A$  are moment maps for  $l, r$ , such that

- (i)  $\mu_l, \mu_r$  are embeddings, and their images Poisson commute;
- (ii) There exists an  $l(H) \times r(H)$ -invariant subspace  $A_0^l$  of  $A$  such that the multiplication map  $\mu_r(\mathcal{O}(U)) \otimes A_0^l \rightarrow A$  is a linear isomorphism; there exists an  $l(H) \times r(H)$ -invariant subspace  $A_0^r$  of  $A$  such that the multiplication map  $\mu_l(\mathcal{O}(U)) \otimes A_0^r \rightarrow A$  is a linear isomorphism.

A morphism of  $H$ -biequivariant Poisson algebras over  $U$  is a morphism of Poisson algebras which preserves  $l, r$  and  $\mu_l, \mu_r$ .

**Remark 1.** Form  $[l, r] = 0$  it follows that  $\{\mu_l \circ x, \mu_r \circ y\}$  is a central element (in the Poisson sense) for any  $x, y \in \mathfrak{h}$ , but it does not follow that this commutator equals to zero. So we require that it is zero by condition (i).

**Remark 2.** Condition (ii) is of technical nature, and is not very important in the discussion below.

Denote the category of  $H$ -biequivariant Poisson algebras over  $U$  by  $\mathcal{A}_U$ .

For convenience we will write  $l(h)a$  as  $ha$  and  $r(h)a$  as  $ah$ .

Let us now describe the monoidal structure on  $\mathcal{A}_U$ .

Let  $A, B \in \mathcal{A}_U$ . Let  $l_A, r_A, l_B, r_B, \mu_l^A, \mu_r^A, \mu_l^B, \mu_r^B$  be the corresponding actions and moment maps. Consider the action of the group  $H$  in  $A \otimes B$  by  $\Delta(h)(a \otimes b) = ah^{-1} \otimes hb$ . We will construct a new  $H$ -biequivariant Poisson algebra  $A \tilde{\otimes} B$ , which is obtained by Hamiltonian reduction of  $A \otimes B$  by this action of  $H$ .

Denote by  $A * B$  the product  $A \otimes_{\mathcal{O}(U)} B$ , where  $\mathcal{O}(U)$  is mapped to  $A$  via  $\mu_r^A$  and to  $B$  via  $\mu_l^B$ . The algebra  $A * B$  has two commuting actions of  $H$  ( $l_A \otimes 1$  and  $1 \otimes r_B$ ). But we cannot claim that  $A * B \in \mathcal{A}_U$ , since the Poisson structure on  $A \otimes B$  does not, in general, descend to  $A * B$ .

However, the action  $\Delta$  of  $H$  on  $A \otimes B$  descends to one on  $A * B$ , so we can define  $A \tilde{\otimes} B := (A * B)^H$ , where  $H$  acts by  $\Delta$ . It is easy to check that the Poisson structure on  $A \otimes B$  descends to one on  $A \tilde{\otimes} B$  (Hamiltonian reduction). The two actions of  $H$  and their moment maps also descend to  $A \tilde{\otimes} B$ . So, in order to check that  $A \tilde{\otimes} B \in \mathcal{A}_U$ , it suffices to check properties (i) and (ii).

Using properties (i) and (ii) of the moment maps  $\mu_l^A, \mu_r^A, \mu_l^B, \mu_r^B$ , it is easy to see that  $A * B$  is naturally identified with  $\mu_l^A(\mathcal{O}(U)) \otimes A_0^r \otimes B_0^r$ , and  $A \tilde{\otimes} B$  is identified with  $\mu_l^A(\mathcal{O}(U)) \otimes (A_0^r \otimes B_0^r)^H$ , where  $H$  acts by  $a \otimes b \rightarrow ah^{-1} \otimes hb$ . This implies properties (i) and (ii) for the moment map  $\mu_l^A \otimes 1 : \mathcal{O}(U) \rightarrow A \tilde{\otimes} B$ , corresponding to the left action of  $H$  on  $A \tilde{\otimes} B$  (with  $(A \tilde{\otimes} B)_0^r = (A_0^r \otimes B_0^r)^H$ ). For the moment map  $1 \otimes \mu_r^B : \mathcal{O}(U) \rightarrow A \tilde{\otimes} B$  corresponding to the right action, these properties are proved analogously.

Thus,  $A \tilde{\otimes} B \in \mathcal{A}_U$ . It is clear that the assignment  $A, B \rightarrow A \tilde{\otimes} B$  is a bifunctor  $\mathcal{A}_U \times \mathcal{A}_U \rightarrow \mathcal{A}_U$ .

Consider the algebra  $\mathcal{O}((T^*H)_U)$ , with the standard Poisson structure, equipped with the standard actions  $l, r$  of  $H$  on left and right given by  $(x, p) \rightarrow (h_1 x h_2, h_1 p h_2)$ . Let  $M_l, M_r : (T^*H)_U \rightarrow U$  be given by  $(h, p) \rightarrow ph^{-1}$ ,  $(h, p) \rightarrow h^{-1}p$ . Let  $\mu_{l,r} = M_{l,r}^* : \mathcal{O}(U) \rightarrow \mathcal{O}((T^*H)_U)$ . It is easy to check that  $\mu_{l,r}$  are moment maps for  $l, r$ .

Let  $\mathbf{1} = (\mathcal{O}((T^*H)_U), l, r, \mu_l, \mu_r) \in \mathcal{A}_U$ . It is easy to check that we have natural isomorphisms  $A \tilde{\otimes} \mathbf{1} \equiv A \equiv \mathbf{1} \tilde{\otimes} A$ .

**Proposition 2.2.** (i)  $(A \tilde{\otimes} B) \tilde{\otimes} C = A \tilde{\otimes} (B \tilde{\otimes} C)$ .

(ii)  $\mathbf{1}$  is a unit object in  $\mathcal{A}_U$  with respect to  $\tilde{\otimes}$ , and  $(\mathcal{A}_U, \tilde{\otimes}, \mathbf{1})$  is a monoidal category.

*Proof.* Easy.

Let  $A \in \mathcal{A}_U$ . Denote by  $\bar{A}$  the new object of  $\mathcal{A}_U$  obtained as follows:  $\bar{A}$  is  $A$  as an algebra, with the opposite Poisson structure, the left and the right actions of  $H$  permuted (i.e. the left, respectively right, action of  $h$  on  $\bar{A}$  is the right, respectively left, action of  $h^{-1}$  on  $A$ ), and the moment maps also permuted. We will call  $\bar{A}$  the dual object to  $A$ . By a *reflection* on  $A$  we will mean an involutive morphism  $i : \bar{A} \rightarrow A$ .

Let  $A \in \mathcal{A}_U$  and  $i : \bar{A} \rightarrow A$  be a reflection. Let  $\varphi_+^i, \varphi_-^i : A \otimes A \rightarrow A$  be given by the formulas  $\varphi_+^i(a \otimes b) = ai(b)$ ,  $\varphi_-^i(a \otimes b) = i(a)b$ . It is easy to see that these maps descend to maps (not necessarily Poisson)  $\psi_{\pm}^i : A \tilde{\otimes} A \rightarrow A$ .

## 2.4. $H$ -biequivariant Poisson-Hopf algebroids.

Now let us define the algebraic version of the notion of an  $H$ -biequivariant Poisson groupoid – the notion of an  $H$ -biequivariant Poisson-Hopf algebroid.

**Definition.** Let  $A$  be an  $H$ -biequivariant Poisson algebra. Then  $A$  is called an  $H$ -biequivariant Poisson-Hopf algebroid over  $U$  if it is equipped with a coassociative  $\mathcal{A}_U$ -morphism  $\Delta : A \rightarrow A \tilde{\otimes} A$  called the coproduct, an  $\mathcal{A}_U$ -morphism  $\varepsilon : A \rightarrow \mathbf{1}$  called the counit, and a reflection  $S : \bar{A} \rightarrow A$  called the antipode, such that

(i)  $(id \bullet \varepsilon) \circ \Delta = (\varepsilon \bullet id) \circ \Delta = id$ , and

(ii)  $\psi_+^S \circ \Delta = \mu_l \circ P \circ \varepsilon$ ,  $\psi_-^S \circ \Delta = \mu_r \circ P \circ \varepsilon$ , where  $P : \mathbf{1} \rightarrow \mathcal{O}(U)$  acts by  $f(x, p) \rightarrow f(1, p)$ .

**Remark.** In the above discussion,  $U$  is a Zariski open set. If  $k = \mathbb{R}$  or  $\mathbb{C}$ , then we can take  $U$  to be an open set in the usual sense, and define  $\mathcal{O}(U)$  to be the algebra of smooth, respectively analytic, functions on  $U$ . Then we can repeat sections 2.3, 2.4 and thus define the notions of an  $H$ -biequivariant Poisson algebra and Poisson-Hopf algebroid over  $U$ . Similarly, one can take  $U$  to be the infinitesimal neighborhood of zero in  $\mathfrak{h}^*$  (i.e.  $\mathcal{O}(U) = k[[\mathfrak{h}]]$ ). The material of Sections 2.3 and 2.4 can also be generalized to this case in a straightforward way.

The constructions of Section 1.3 can easily be put in the algebraic framework of Sections 2.3, 2.4. Let  $G$  be an affine algebraic group, and  $H$  a connected algebraic subgroup of  $G$ . Let  $X(G, H, U)$  be a coboundary dynamical Poisson groupoid defined by (1.1), (1.4). Consider the algebra  $A = \mathcal{O}(U) \otimes \mathcal{O}(G) \otimes \mathcal{O}(U)$ , where  $\mathcal{O}(G)$  is the algebra of polynomial functions on  $G$ . It is easy to see that  $A$  is closed under the Poisson bracket defined by (1.1), (1.4), so it is a Poisson algebra. The two actions of  $H$  on  $A$  are defined by

$$(2.1) \quad \begin{aligned} l(h)[a \otimes f \otimes b](u_1, g, u_2) &= [a \otimes f \otimes b](h^{-1}u_1h, h^{-1}g, u_2), \\ r(h)[a \otimes f \otimes b](u_1, g, u_2) &= [a \otimes f \otimes b](u_1, gh^{-1}, hu_2h^{-1}), \end{aligned}$$

and the corresponding moment maps are the maps corresponding to the projections of  $X$  to the first and second component of  $U$ . The coproduct, counit, and antipode in  $A$  are defined by the groupoid structure on  $X$ . Thus,  $A$  is an  $H$ -biequivariant Poisson-Hopf algebroid. We will call such a  $H$ -biequivariant Poisson-Hopf algebroid a dynamical Poisson-Hopf algebroid.

The notion of a dynamical Poisson-Hopf algebroid will be useful to us in the next paper.

### 3. CLASSIFICATION OF CLASSICAL DYNAMICAL $r$ -MATRICES

#### 3.1. Kac-Moody algebras [K, Ch. 2].

Let  $A = (a_{i,j})_{i,j=1}^n$  be a symmetrizable generalized Cartan matrix. Let  $\mathfrak{g}(A)$  be the associated Kac-Moody Lie algebra,  $\mathfrak{h}$  its Cartan subalgebra,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

the root decomposition of the Kac-Moody Lie algebra. Let  $(\cdot, \cdot)$  be an invariant nondegenerate bilinear form on  $\mathfrak{g}$  and  $\Omega \in \mathfrak{g} \hat{\otimes} \mathfrak{g}$  the associated Casimir operator. Here  $\mathfrak{g} \hat{\otimes} \mathfrak{g}$  denotes the completed tensor product.

**Remark.** In this work we consider the Kac-Moody Lie algebras associated with a symmetrizable generalized Cartan matrix, although all theorems with the same proof are valid for more general Kac-Moody Lie algebras associated with a symmetric complex matrix, see [SV], [V, 11.1.10].

#### 3.2. Classical dynamical $r$ -matrices.

Let  $\mathfrak{g}$  be a Kac-Moody Lie algebra,  $\mathfrak{h}$  its Cartan subalgebra. A meromorphic function

$$r : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$$

is called a *classical dynamical r-matrix associated with the pair  $\mathfrak{h} \subset \mathfrak{g}$*  if it satisfies the following three conditions.

1. *The zero weight condition,*

$$(3.1) \quad [h \otimes 1 + 1 \otimes h, r(\lambda)] = 0$$

for any  $\lambda \in \mathfrak{h}^*$  and  $h \in \mathfrak{h}$ .

2. *The generalized unitarity,*

$$(3.2) \quad r^{12}(\lambda) + r^{21}(\lambda) = \epsilon \Omega$$

for some constant  $\epsilon \in \mathbb{C}$  and all  $\lambda$ .

3. *The classical dynamical Yang-Baxter equation, CDYB,*

$$(3.3) \quad \text{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

Here we use the following notations and conventions. By a meromorphic function  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$  satisfying the zero weight condition we mean a function of the form  $r = r_{\mathfrak{h}} + \sum_{\alpha \in \Delta} r_{\alpha}$  where  $\Delta$  is the set of roots and  $r_{\mathfrak{h}} : \mathfrak{h}^* \rightarrow \mathfrak{h} \otimes \mathfrak{h}$ ,  $r_{\alpha} : \mathfrak{h}^* \rightarrow \mathfrak{g}_{\alpha} \otimes \mathfrak{g}_{-\alpha}$  are meromorphic maps of finite dimensional spaces.

If  $X \in \text{End}(V_i)$ , then we denote by  $X^{(i)} \in \text{End}(V_1 \otimes \cdots \otimes V_n)$  the operator  $\cdots \otimes \text{Id} \otimes X \otimes \text{Id} \otimes \cdots$ , acting non-trivially on the  $i$ th factor of a tensor product of vector spaces, and if  $X = \sum X_k \otimes Y_k \in \text{End}(V_i \otimes V_j)$ , then we set  $X^{ij} = \sum X_k^{(i)} Y_k^{(j)}$ .

The differential of the r-matrix is considered in (3.3) as a meromorphic function

$$dr : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g} \hat{\otimes} \mathfrak{g}, \quad \lambda \mapsto \sum_i x_i \otimes \frac{\partial r}{\partial x_i}(\lambda),$$

where  $\{x_i\}$  is any basis in  $\mathfrak{h}$ . We denote by  $\text{Alt}(dr)$  the following symmetrization of  $dr$  with respect to even permutations of 1, 2, 3,

$$\text{Alt}(dr) = \sum_i x_i^{(1)} \frac{\partial r^{23}}{\partial x_i} + \sum_i x_i^{(2)} \frac{\partial r^{31}}{\partial x_i} + \sum_i x_i^{(3)} \frac{\partial r^{12}}{\partial x_i}.$$

The CDYB equation is an equation in  $\mathfrak{g} \hat{\otimes} \mathfrak{g} \hat{\otimes} \mathfrak{g}$ .

The constant  $\epsilon$  in (3.2) is called *the coupling constant*.

### 3.3. Classification of the classical dynamical r-matrices with nonzero coupling constant.

#### Theorem 3.1.

1. Let  $x_i, i = 1, \dots, N$ , be a basis in  $\mathfrak{h}$ . For any positive root  $\alpha \in \Delta_+$  let  $e_{\alpha}^i, i = 1, \dots, N_{\alpha}$ , be a basis in the root space  $\mathfrak{g}_{\alpha}$  and  $e_{-\alpha}^i, i = 1, \dots, N_{\alpha}$ , the dual basis in  $\mathfrak{g}_{-\alpha}$ . Let  $\nu$  be an element in  $\mathfrak{h}^*$ ,  $C = \sum_{i,j} C_{i,j} dx_i \otimes dx_j$  a closed meromorphic 2-form on  $\mathfrak{h}^*$ ,  $\epsilon$  a nonzero complex number. Then the function  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$  defined by

$$(3.4) \quad r(\lambda) = \sum_{i,j=1}^N C_{i,j}(\lambda) x_i \otimes x_j + \frac{\epsilon}{2} \Omega + \sum_{\alpha \in \Delta} \sum_{i=1}^{N_{\alpha}} \frac{\epsilon}{2} \cotanh\left(\frac{\epsilon}{2}(\alpha, \lambda - \nu)\right) e_{\alpha}^i \otimes e_{-\alpha}^i$$



is a classical dynamical  $r$ -matrix with the coupling constant  $\epsilon$ . Here  $\text{cotanh}$  is the hyperbolic cotangent.

2. Let  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$  be a classical dynamical  $r$ -matrix with nonzero coupling constant,  $r = r_{\mathfrak{h}} + \sum_{\alpha \in \Delta} r_{\alpha}$  its weight decomposition. If the function  $r_{\alpha}$  is not constant for any simple positive root  $\alpha$ , then the function  $r$  has the form indicated in (3.4).

Theorem 3.1 is proved in Section 3.6.

For a simple Lie algebra  $\mathfrak{g}$ , the complete classification of  $r$ -matrices  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$  with nonzero coupling constant is given in Section 3.7.

### 3.4. Classification of the classical dynamical $r$ -matrices with zero coupling constant.

We shall use the notations of Theorem 3.1.

#### Theorem 3.2.

1. Let  $X$  be a subset of the set of roots,  $\Delta$ , of a Kac-Moody Lie algebra  $\mathfrak{g}$  such that
  - 1) if  $\alpha, \beta \in X$  and  $\alpha + \beta$  is a root then  $\alpha + \beta \in X$ , and
  - 2) if  $\alpha \in X$  then  $-\alpha \in X$ .

Let  $\nu$  be an element in  $\mathfrak{h}^*$ ,  $C = \sum_{i,j} C_{i,j} dx_i \otimes dx_j$  a closed meromorphic 2-form on  $\mathfrak{h}^*$ . Then the function  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$  defined by

$$(3.5) \quad r(\lambda) = \sum_{i,j=1}^N C_{i,j}(\lambda) x_i \otimes x_j + \sum_{\alpha \in X} \sum_{i=1}^{N_{\alpha}} \frac{1}{(\alpha, \lambda - \nu)} e_{\alpha}^i \otimes e_{-\alpha}^i$$

is a classical dynamical  $r$ -matrix with zero coupling constant.

2. If  $\mathfrak{g}$  is a simple Lie algebra, then any classical dynamical  $r$ -matrix with zero coupling constant has this form.
3. Let  $\mathfrak{g}$  be an arbitrary Kac-Moody Lie algebra. Let  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$  be a classical dynamical  $r$ -matrix with zero coupling constant,  $r = r_{\mathfrak{h}} + \sum_{\alpha \in \Delta} r_{\alpha}$  its weight decomposition. If the function  $r_{\alpha}$  is not identically equal to zero for any simple positive root  $\alpha$ , then the function  $r$  has the form indicated in (3.5) with  $X = \Delta$ .

Theorem 3.2 is proved in Section 3.5.

**Example.** Let  $\mathfrak{g}$  be the Lie algebra of type  $B_2$  with roots  $(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)$ . Then the set of long roots  $(\pm 1, \pm 1)$  gives an example of the set  $X$ .

Consider an  $r$ -matrix of type (3.5). Assume that the element  $\nu \in \mathfrak{h}^*$  tends to infinity so that all terms of the matrix have limit. Then the limiting function is an  $r$ -matrix of type (3.5) for a new set  $X$ .

Notice that the  $r$ -matrix (3.5) corresponding to the example is not a limiting case of the  $r$ -matrix (3.5) with  $X = \Delta$ .

### 3.5. Proof of Theorem 3.2.

First we prove Theorem 3.2 assuming that  $\mathfrak{g}$  is a simple Lie algebra. In this case  $\dim \mathfrak{g}_{\alpha} = 1$  for any root  $\alpha$ . For any positive root  $\alpha$  fix elements  $e_{\alpha} \in \mathfrak{g}_{\alpha}$  and  $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$  dual with respect to the bilinear form.

Let  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  be a classical dynamical  $r$ -matrix with zero coupling constant. The zero weight condition and the unitarity condition imply that the  $r$ -matrix could be written in the form

$$r(\lambda) = \sum_{i,j=1}^N C_{i,j}(\lambda) x_i \otimes x_j + \sum_{\alpha \in \Delta} \varphi_{\alpha}(\lambda) e_{\alpha} \otimes e_{-\alpha}$$

where  $\varphi_\alpha, C_{i,j}$  are suitable scalar meromorphic functions such that  $\varphi_{-\alpha}(\lambda) = -\varphi_\alpha(\lambda)$ , and  $C_{i,j}(\lambda) = -C_{j,i}(\lambda)$ .

The CDYB equation is an equation in  $\mathfrak{g}^{\otimes 3}$ . The unitarity condition implies that the left hand side of the CDYB equation is skew-symmetric with respect to permutations of factors. This remark and the zero weight condition show that in order to solve the CDYB equation it is enough to solve its  $\mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h}-$ ,  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}-$  and  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_\gamma-$ parts, where  $\alpha, \beta, \gamma \in \Delta$  and in the last case  $\alpha + \beta + \gamma = 0$ .

A basis in  $\mathfrak{g}^{\otimes 3}$  is formed by the elements  $x \otimes y \otimes z$  where  $x, y, z$  run through  $x_i, e_\alpha, i = 1, \dots, N, \alpha \in \Delta$ .

The  $x_i \otimes x_j \otimes x_k$ -part of the CDYB equation has the form

$$(3.6) \quad \frac{\partial C_{j,k}}{\partial x_i} + \frac{\partial C_{k,i}}{\partial x_j} + \frac{\partial C_{i,j}}{\partial x_k} = 0$$

and says that  $\sum_{i,j=1}^N C_{i,j}(\lambda) dx_i \otimes dx_j$  is a closed differential form.

The  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ -part of the CDYB equation has the form

$$\sum_k \frac{\partial \varphi_\alpha}{\partial x_k} x_k \otimes e_\alpha \otimes e_{-\alpha} + \varphi_\alpha^2 h_\alpha \otimes e_\alpha \otimes e_{-\alpha} = 0$$

where  $h_\alpha = [e_\alpha, e_{-\alpha}]$ . This equation could be written in the form

$$(3.7) \quad d\varphi_\alpha + \varphi_\alpha^2 dh_\alpha = 0.$$

Hence  $\varphi_\alpha = 0$  or  $\varphi_\alpha = (h_\alpha - \nu_\alpha)^{-1}$  for some  $\nu_\alpha \in \mathbb{C}$ . Here  $h_\alpha$  is considered as a linear function on  $\mathfrak{h}^*$ ,  $h_\alpha(\lambda) = (\alpha, \lambda)$  for  $\lambda \in \mathfrak{h}^*$ .

The  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_{-\alpha-\beta}$ -part of the CDYB equation has the form

$$(3.8) \quad \varphi_\alpha \varphi_\beta - \varphi_\alpha \varphi_{\alpha+\beta} - \varphi_{\alpha+\beta} \varphi_\beta = 0$$

**Lemma 3.3.** *Let  $X = \{\alpha \in \Delta \mid \varphi_\alpha \neq 0\}$ . Then  $X$  is closed with respect to multiplication by  $-1$  and addition.*

*Proof.* The set  $X$  is closed with respect to multiplication by  $-1$  because of the unitarity condition. If  $\varphi_\alpha$  and  $\varphi_\beta$  are different from zero, then  $\varphi_{\alpha+\beta}(\varphi_\alpha + \varphi_\beta) = \varphi_\alpha \varphi_\beta$ , and, hence,  $\varphi_{\alpha+\beta}$  is different from zero.  $\square$

**Lemma 3.4.**

$$(3.9) \quad \nu_{\alpha+\beta} = \nu_\alpha + \nu_\beta.$$

*Proof.* Equation (3.8) implies

$$\frac{1}{h_\alpha - \nu_\alpha} \frac{1}{h_\beta - \nu_\beta} = \left( \frac{1}{h_\alpha - \nu_\alpha} + \frac{1}{h_\beta - \nu_\beta} \right) \frac{1}{h_{\alpha+\beta} - \nu_{\alpha+\beta}}.$$

Since  $h_{\alpha+\beta} = h_\alpha + h_\beta$ , in order to cancel the last pole one needs (3.9).  $\square$

**Corollary 3.5.** *There is  $\nu \in \mathfrak{h}^*$  such that  $\nu_\alpha = (\alpha, \nu)$  for all  $\alpha \in X$ .*

This finishes the proof of Theorem 3.2 for a simple Lie algebra.

Now we assume that  $\mathfrak{g}$  is an arbitrary Kac-Moody algebra. Let  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$  be a classical dynamical r-matrix with zero coupling constant. The zero weight condition and the unitarity condition imply that the r-matrix could be written in the form

$$r(\lambda) = \sum_{i,j=1}^N C_{i,j}(\lambda) x_i \otimes x_j + \sum_{\alpha \in \Delta} \sum_{i,j=1}^{N_\alpha} \varphi_\alpha^{i,j}(\lambda) e_\alpha^i \otimes e_{-\alpha}^j$$

where  $C_{i,j}(\lambda)$ ,  $\varphi_\alpha^{i,j}(\lambda)$  are suitable scalar functions such that  $C_{i,j} = -C_{j,i}$  and  $\varphi_{-\alpha}^{j,i} = -\varphi_\alpha^{i,j}$ .

The  $\mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h}$ -part of the CDYB equation is the same as for a simple Lie algebra and means that  $\sum_{i,j=1}^N C_{i,j}(\lambda) dx_i \otimes dx_j$  is a closed differential form.

To analyze the  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ - and  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_{-\alpha-\beta}$ -parts of the CDYB equation we shall introduce some useful linear operators  $\varphi_\alpha(\lambda) : \mathfrak{g}_\alpha \rightarrow \mathfrak{g}_\alpha$ , where  $\alpha \in \Delta$  and  $\lambda \in \mathfrak{h}^*$ . Namely, for any  $\alpha \in \Delta$  and  $\lambda \in \mathfrak{h}^*$ , we define  $\varphi_\alpha(\lambda)$  by the formula

$$(3.10) \quad \varphi_\alpha(\lambda) : e_\alpha^j \mapsto \sum_{i=1}^{N_\alpha} \varphi_\alpha^{i,j}(\lambda) e_\alpha^i.$$

Let  $r_\alpha(\lambda)$  denote the  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ -part of the r-matrix,

$$r_\alpha(\lambda) = \sum_{i,j=1}^{N_\alpha} \varphi_\alpha^{i,j}(\lambda) e_\alpha^i \otimes e_{-\alpha}^j = \sum_{j=1}^{N_\alpha} \varphi_\alpha(\lambda) e_\alpha^j \otimes e_{-\alpha}^j.$$

**Lemma 3.6.** *The  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ -part of the CDYB equation has the form*

$$(3.11) \quad d\varphi_\alpha + \varphi_\alpha^2 dh_\alpha = 0.$$

where the operator valued function  $\varphi_\alpha : \mathfrak{h}^* \rightarrow \text{End}(\mathfrak{g}_\alpha)$  is defined in (3.10).

*Proof.* The  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ -part of the CDYB equation has the form

$$\sum_{i,j=1}^{N_\alpha} \sum_k \frac{\partial \varphi_\alpha^{i,j}}{\partial x_k}(\lambda) x_k \otimes e_\alpha^i \otimes e_{-\alpha}^j + [r_{-\alpha}^{12}(\lambda), r_\alpha^{13}(\lambda)] = 0.$$

To prove the Lemma it is enough to show that

$$[r_{-\alpha}^{12}(\lambda), r_\alpha^{13}(\lambda)] = \sum_{l=1}^{N_\alpha} h_\alpha \otimes (\varphi_\alpha)^2 e_\alpha^l \otimes e_{-\alpha}^l.$$

We shall use the formula

$$[e_\alpha^i, e_{-\alpha}^j] = (e_\alpha^i, e_{-\alpha}^j) h_\alpha = \delta_{i,j} h_\alpha.$$

Then

$$\begin{aligned}
[r_{-\alpha}^{12}(\lambda), r_{\alpha}^{13}(\lambda)] &= [\sum_{i,k} \varphi_{-\alpha}^{i,k} e_{-\alpha}^i \otimes e_{\alpha}^k \otimes 1, \sum_{j,l} \varphi_{\alpha}^{j,l} e_{\alpha}^j \otimes 1 \otimes e_{-\alpha}^l] = \\
&\sum_{i,k,j,l} \varphi_{-\alpha}^{i,k} \varphi_{\alpha}^{j,l} [e_{-\alpha}^i, e_{\alpha}^j] \otimes e_{\alpha}^k \otimes e_{-\alpha}^l = \sum_{i,k,l} \varphi_{\alpha}^{k,i} \varphi_{\alpha}^{i,l} h_{\alpha} \otimes e_{\alpha}^k \otimes e_{-\alpha}^l = \\
&\sum_l h_{\alpha} \otimes (\varphi_{\alpha})^2 e_{\alpha}^l \otimes e_{-\alpha}^l.
\end{aligned}$$

The Lemma is proved.  $\square$

Introduce a linear map

$$\Lambda : \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}, \quad x \otimes y \otimes z \mapsto (x, [y, z]).$$

Recall that the invariance of the bilinear form implies  $(x, [y, z]) = ([x, y], z)$ .

**Lemma 3.7.**

Let  $\alpha, \beta, \gamma \in \Delta$  be any roots such that  $\alpha + \beta + \gamma = 0$ . Then the  $\mathfrak{g}_{-\alpha} \otimes \mathfrak{g}_{-\beta} \otimes \mathfrak{g}_{-\gamma}$ -part of the CDYB equation is equivalent to the statement that the composition map

$$(3.12) \quad \Lambda \circ (\varphi_{\alpha} \otimes \varphi_{\beta} \otimes 1 + \varphi_{\alpha} \otimes 1 \otimes \varphi_{\gamma} + 1 \otimes \varphi_{\beta} \otimes \varphi_{\gamma}) : \mathfrak{g}_{\alpha} \otimes \mathfrak{g}_{\beta} \otimes \mathfrak{g}_{\gamma} \rightarrow \mathbb{C}$$

is the zero map.

*Proof.* The  $\mathfrak{g}_{-\alpha} \otimes \mathfrak{g}_{-\beta} \otimes \mathfrak{g}_{-\gamma}$ -part has the form

$$[r_{-\alpha}^{13}(\lambda), r_{-\beta}^{23}(\lambda)] + [r_{-\alpha}^{12}(\lambda), r_{\gamma}^{13}(\lambda)] + [r_{-\alpha}^{12}(\lambda), r_{\gamma}^{23}(\lambda)] = 0.$$

Compute each of the terms.

$$\begin{aligned}
[r_{-\alpha}^{13}(\lambda), r_{-\beta}^{23}(\lambda)] &= [\sum_{i,j} \varphi_{-\alpha}^{i,j} e_{-\alpha}^i \otimes 1 \otimes e_{\alpha}^j, \sum_{k,l} \varphi_{-\beta}^{k,l} 1 \otimes e_{-\beta}^k \otimes e_{\beta}^l] = \\
&\sum_{i,j,k,l} \varphi_{-\alpha}^{i,j} \varphi_{-\beta}^{k,l} e_{-\alpha}^i \otimes e_{-\beta}^k \otimes [e_{\alpha}^j, e_{\beta}^l] = \sum_{i,j,k,l,m} \varphi_{-\alpha}^{i,j} \varphi_{-\beta}^{k,l} e_{-\alpha}^i \otimes e_{-\beta}^k \otimes e_{-\gamma}^m \cdot (e_{\gamma}^m, [e_{\alpha}^j, e_{\beta}^l]).
\end{aligned}$$

Using the equalities  $\sum_j \varphi_{-\alpha}^{i,j} e_{\alpha}^j = -\varphi_{\alpha} e_{\alpha}^i$ ,  $\sum_l \varphi_{-\beta}^{k,l} e_{\beta}^l = -\varphi_{\beta} e_{\beta}^k$ ,  $(e_{\gamma}^m, [e_{\alpha}^j, e_{\beta}^l]) = (e_{\alpha}^j, [e_{\beta}^l, e_{\gamma}^m])$ , we get

$$[r_{-\alpha}^{13}(\lambda), r_{-\beta}^{23}(\lambda)] = \sum_{i,k,m} \Lambda(\varphi_{\alpha} e_{\alpha}^i \otimes \varphi_{\beta} e_{\beta}^k \otimes e_{\gamma}^m) e_{-\alpha}^i \otimes e_{-\beta}^k \otimes e_{-\gamma}^m.$$

Similarly

$$\begin{aligned}
[r_{\beta}^{12}(\lambda), r_{\gamma}^{13}(\lambda)] &= \sum_{i,k,m} \Lambda(e_{\alpha}^i \otimes \varphi_{\beta} e_{\beta}^k \otimes \varphi_{\gamma} e_{\gamma}^m) e_{-\alpha}^i \otimes e_{-\beta}^k \otimes e_{-\gamma}^m, \\
[r_{-\alpha}^{12}(\lambda), r_{\gamma}^{13}(\lambda)] &= \sum_{i,k,m} \Lambda(\varphi_{\alpha} e_{\alpha}^i \otimes e_{\beta}^k \otimes \varphi_{\gamma} e_{\gamma}^m) e_{-\alpha}^i \otimes e_{-\beta}^k \otimes e_{-\gamma}^m.
\end{aligned}$$

These formulae prove the Lemma.  $\square$

Lemmas 3.6 and 3.7 imply the first statement of Theorem 3.2 for an arbitrary Kac-Moody Lie algebra.

We shall show the third statement of Theorem 3.2 by induction on the complexity of the roots. Let  $\alpha_1, \dots, \alpha_r$  be the simple positive roots of  $\mathfrak{g}$  and  $\alpha \in \Delta$  any root. Represent it as a linear combination of the simple roots,  $\alpha = k_1 \alpha_1 + \dots + k_r \alpha_r$ . The absolute value of the number  $k_1 + \dots + k_r$  will be called the complexity of the root  $\alpha$ . Thus the only roots of complexity one are the simple roots.

**Lemma 3.8.**

Let  $\mathfrak{g}$  be a Kac-Moody Lie algebra. Let  $r = r_{\mathfrak{h}} + \sum_{\alpha \in \Delta} r_{\alpha}$  be a classical dynamical  $r$ -matrix with zero coupling constant and with nonzero components  $r_{\alpha}$  corresponding to the simple positive roots. Then the associated operator valued functions  $\varphi_{\alpha} : \mathfrak{h}^* \rightarrow \text{End}(\mathfrak{g}_{\alpha})$  have the following form. There exists  $\nu \in \mathfrak{h}^*$  such that

$$(3.13) \quad \varphi_{\alpha}(\lambda) = \frac{1}{(\alpha, \lambda - \nu)} \text{id}$$

for all  $\alpha \in \Delta$  and  $\lambda \in \mathfrak{h}^*$ .

*Proof.* For any simple root  $\alpha$ ,  $\dim \mathfrak{g}_{\alpha} = 1$ . Then  $\varphi_{\alpha}$  is a scalar function of the form  $\varphi_{\alpha}(\lambda) = (h_{\alpha} - \nu_{\alpha})^{-1}$  for some  $\nu_{\alpha} \in \mathbb{C}$ , see Lemma 3.6. Hence there exists  $\nu \in \mathfrak{h}^*$  such that the operator  $\varphi_{\alpha}(\lambda)$  has the form indicated in (3.13) for all simple roots  $\alpha$ . Our goal is to extend this formula to all roots.

Let  $e_1, \dots, e_r, f_1, \dots, f_r$  be the Chevalley generators corresponding to the simple positive roots  $\alpha_1, \dots, \alpha_r$ . For any root  $\alpha$ , the space  $\mathfrak{g}_{\alpha}$  is generated by commutators of the elements  $e_1, \dots, e_r$ , if  $\alpha$  is positive, and by commutators of the elements  $f_1, \dots, f_r$ , if  $\alpha$  is negative. Hence for any root  $\alpha$  the space  $\mathfrak{g}_{\alpha}$  is generated by commutators of the form  $[x, y]$  where  $x \in \mathfrak{g}_{\beta}$ ,  $y \in \mathfrak{g}_{\gamma}$ ,  $\alpha = \beta + \gamma$ , and the complexity of  $\beta$  and  $\gamma$  is less than the complexity of  $\alpha$ .

Assume now that a root  $\gamma$  has the following property. For any roots  $\alpha$  and  $\beta$  such that  $\alpha + \beta + \gamma = 0$  and the complexity of  $\alpha$  and  $\beta$  is less than the complexity of  $\gamma$ , the operators  $\varphi_{\alpha}$  and  $\varphi_{\beta}$  have the form indicated in (3.13). Let us show that the operator  $\varphi_{\gamma}$  has the same form.

Formula (3.12) implies

$$\left( \frac{1}{(\alpha, \lambda - \nu)} + \frac{1}{(\beta, \lambda - \nu)} \right) ([x, y], \varphi_{\gamma}(\lambda) z) = - \frac{1}{(\alpha, \lambda - \nu)(\beta, \lambda - \nu)} ([x, y], z).$$

This formula means that the operator  $\varphi_{\gamma}(\lambda)$  is uniquely determined by the operators  $\varphi_{\alpha}$  and  $\varphi_{\beta}$  corresponding to the roots  $\alpha$  and  $\beta$  with the complexity less than the complexity of  $\gamma$ . At the same time we see that if the operator  $\varphi_{\gamma}$  has the form indicated in (3.13), then it satisfies both equations (3.12) and (3.11). Lemma 3.8 and Theorem 3.2 are proved.  $\square$

### 3.6. Proof of Theorem 3.1.

First we prove Theorem 3.1 assuming that  $\mathfrak{g}$  is a simple Lie algebra. As in the proof of Theorem 3.2, fix the dual generators  $e_{\alpha} \in \mathfrak{g}_{\alpha}$  and  $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$  for all roots  $\alpha \in \Delta$ . Fix an orthonormal basis in  $\mathfrak{h}$ ,  $x_1, \dots, x_N$ .

Let  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  be a meromorphic map,  $\epsilon$  a nonzero complex number,  $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$  the Casimir operator of  $\mathfrak{g}$  associated to the bilinear form,  $\Omega = \sum_k x_k \otimes x_k + \sum_{\alpha \in \Delta} e_{\alpha} \otimes e_{-\alpha}$ .

Introduce a meromorphic map  $s : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  by the formula  $s(\lambda) = r(\lambda) - \epsilon \Omega / 2$ .

**Lemma 3.9.** *The map  $r$  is a classical dynamical  $r$ -matrix with the coupling constant  $\epsilon$  if and only if the map  $s$  satisfies the zero weight condition (3.1), the unitarity condition,*

$$s^{12}(\lambda) + s^{21}(\lambda) = 0,$$

and the following analog of the CDYB equation,

$$(3.14) \quad \begin{aligned} & \text{Alt}(ds) + [s^{12}, s^{13}] + [s^{12}, s^{23}] + [s^{13}, s^{23}] \\ & + \frac{\epsilon^2}{4} ([\Omega^{12}, \Omega^{13}] + [\Omega^{12}, \Omega^{23}] + [\Omega^{13}, \Omega^{23}]) = 0. \end{aligned}$$

*Proof.* The only thing that needs to be checked is the fact that the terms of the form  $\epsilon [\Omega^{(i,j)}, s^{(k,l)}]/2$  cancel in the CDYB equation. This can be verified by an easy direct calculation.  $\square$

The zero weight condition and the unitarity condition imply that the matrix  $s$  can be written in the form

$$s(\lambda) = \sum_{i,j=1}^N C_{i,j}(\lambda) x_i \otimes x_j + \sum_{\alpha \in \Delta} \varphi_\alpha(\lambda) e_\alpha \otimes e_{-\alpha}$$

where  $\varphi_\alpha, C_{i,j}$  are suitable scalar meromorphic functions such that  $\varphi_{-\alpha}(\lambda) = -\varphi_\alpha(\lambda)$ , and  $C_{i,j}(\lambda) = -C_{j,i}(\lambda)$ .

The CDYB equation (3.14) is an equation in  $\mathfrak{g}^{\otimes 3}$ . Its left hand side is invariant with respect to even permutations of factors. To solve the CDYB equation it suffices to solve its  $\mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h}$ -,  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ - and  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_\gamma$ -parts, where  $\alpha, \beta, \gamma \in \Delta$  and in the last case  $\alpha + \beta + \gamma = 0$ .

The  $x_i \otimes x_j \otimes x_k$ -part of the CDYB equation has the form indicated in (3.6) and says that  $\sum_{i,j=1}^N C_{i,j}(\lambda) dx_i \otimes dx_j$  is a closed differential form.

The  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ -part of the CDYB equation has the form

$$\begin{aligned} & \sum_k \frac{\partial \varphi_\alpha}{\partial x_k} x_k \otimes e_\alpha \otimes e_{-\alpha} + \varphi_\alpha^2 h_\alpha \otimes e_\alpha \otimes e_{-\alpha} + \frac{\epsilon^2}{4} (-h_\alpha \otimes e_\alpha \otimes e_{-\alpha} + \\ & \sum_k (x_k \otimes [x_k, e_\alpha] \otimes e_{-\alpha} + x_k \otimes e_\alpha \otimes [x_k, e_{-\alpha}])) = 0 \end{aligned}$$

where  $h_\alpha = [e_\alpha, e_{-\alpha}]$ . This equation can be written in the form

$$d\varphi_\alpha + (\varphi_\alpha^2 - \frac{\epsilon^2}{4}) dh_\alpha = 0.$$

Hence

$$\varphi_\alpha(\lambda) = \frac{\epsilon}{2} \cotanh\left(\frac{\epsilon}{2}(h_\alpha - \nu_\alpha)\right)$$

for some  $\nu_\alpha \in \mathbb{C}$ , or  $\varphi_\alpha = \pm \epsilon/2$ . Here  $h_\alpha$  is considered as a linear function on  $\mathfrak{h}^*$ , cf. (3.7).

Let  $\alpha, \beta, \gamma \in \Delta$  be roots such that  $\alpha + \beta + \gamma = 0$ . The  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_\gamma$ -part of the CDYB equation (3.14) has the form

$$(3.15) \quad \varphi_\alpha \varphi_\beta + \varphi_\alpha \varphi_\gamma + \varphi_\gamma \varphi_\beta + \frac{\epsilon^2}{4} = 0.$$

If

$$\varphi_\alpha(\lambda) = \frac{\epsilon}{2} \cotanh\left(\frac{\epsilon}{2}(h_\alpha - \nu_\alpha)\right), \quad \varphi_\beta(\lambda) = \frac{\epsilon}{2} \cotanh\left(\frac{\epsilon}{2}(h_\beta - \nu_\beta)\right)$$

for some  $\nu_\alpha, \nu_\beta \in \mathbb{C}$ , then  $\varphi_\gamma$  is not a constant, so

$$\varphi_\gamma(\lambda) = \frac{\epsilon}{2} \coth\left(\frac{\epsilon}{2}(h_\gamma - \nu_\gamma)\right)$$

for some  $\nu_\gamma \in \mathbb{C}$ . Starting from simple positive roots we conclude that all functions  $\varphi_\alpha(\lambda)$  are not constant. Equation (3.15) also implies

$$\coth\left(\frac{\epsilon}{2}(h_\gamma - \nu_\gamma)\right) + \coth\left(\frac{\epsilon}{2}(h_\alpha + h_\beta - \nu_\alpha - \nu_\beta)\right) = 0.$$

Hence, there exists  $\nu \in \mathfrak{h}^*$  such that  $\nu_\alpha = (\alpha, \nu)$  and

$$\varphi_\alpha(\lambda) = \frac{\epsilon}{2} \coth\left(\frac{\epsilon}{2}(\alpha, \lambda - \nu)\right)$$

for all roots  $\alpha \in \Delta$ . Theorem 3.1 is proved for a simple Lie algebra  $\mathfrak{g}$ .

The generalization of the above proof to the case of a Kac-Moody algebra is word by word parallel to the generalization of the proof of Theorem 3.2 from the simple Lie algebra case to the Kac-Moody algebra case.

### 3.7. Classification of classical dynamical r-matrices with nonzero coupling constant, the simple Lie algebra case.

Let  $\mathfrak{g}$  be a simple Lie algebra,  $\mathfrak{h}$  its Cartan subalgebra,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

the root decomposition,  $(\cdot, \cdot)$  an invariant nondegenerate bilinear form on  $\mathfrak{g}$ . For any positive root  $\alpha \in \Delta(\mathfrak{h})$  fix basis elements  $e_\alpha \in \mathfrak{g}_\alpha$  and  $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$  which are dual with respect to the bilinear form. Fix a basis in the Cartan subalgebra,  $\{x_i\}$ , orthonormal with respect to the bilinear form.

Let

$$\Delta = \Delta_+ \cup \Delta_-$$

be a polarization of roots into positive and negative,  $\Delta_+^s \subset \Delta_+$  the set of simple positive roots. Fix a subset  $X \subset \Delta_+^s$  of the set of positive simple roots.

Fix a nonzero complex number  $\epsilon$  and an element  $\mu \in \mathfrak{h}^*$ . For any root  $\alpha$  introduce a meromorphic function  $\varphi_\alpha : \mathfrak{h}^* \rightarrow \mathbb{C}$  by the following rule. If a root  $\alpha$  is a linear combination of simple roots from  $X$ , then we set

$$\varphi_\alpha(\lambda) = \frac{\epsilon}{2} \coth\left(\frac{\epsilon}{2}(\alpha, \lambda - \mu)\right).$$

Otherwise we set  $\varphi_\alpha(\lambda) = \epsilon/2$ , if  $\alpha$  is positive, and  $\varphi_\alpha(\lambda) = -\epsilon/2$ , if  $\alpha$  is negative.

Let  $C = \sum_{i,j} C_{i,j} dx_i \otimes dx_j$  be a closed meromorphic 2-form on  $\mathfrak{h}^*$ .

#### Theorem 3.10.

1. Introduce a function  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \hat{\otimes} \mathfrak{g}$  by the formula

$$r(\lambda) = \sum_{i,j=1}^N C_{i,j}(\lambda) x_i \otimes x_j + \frac{\epsilon}{2} \Omega + \sum_{\alpha \in \Delta} \varphi_\alpha(\lambda) e_\alpha \otimes e_{-\alpha},$$

where  $C_{i,j}$  and  $\varphi_\alpha$  are defined above. Then  $r$  is a classical dynamical r-matrix with nonzero coupling constant  $\epsilon$ .

2. Any classical dynamical r-matrix,  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ , with nonzero coupling constant has this form.

The proof of Theorem 3.10 is based on the following fact.

**Theorem 3.11.**

Let  $Y \subset \Delta$  be a subset of the set of roots with two properties.

- A. If  $\alpha, \beta \in Y$  and  $\alpha + \beta \in \Delta$ , then  $\alpha + \beta \in Y$ .
  - B. If  $\alpha$  is an element of  $Y$ , then  $-\alpha$  is not an element of  $Y$ .
- Then there exists a polarization  $\Delta = \Delta_+ \cup \Delta_-$  such that  $Y \subset \Delta_+$ .

*Proof of Theorem 3.11.* Consider

$$\mathfrak{n} = \oplus_{\alpha \in Y} \mathbb{C} e_\alpha, \quad \mathfrak{m} = \mathfrak{h} \oplus \oplus_{\alpha \in Y} \mathbb{C} e_\alpha.$$

Then  $\mathfrak{n}, \mathfrak{m}$  are Lie subalgebras of  $\mathfrak{g}$ .

**Lemma 3.12.**

- 1.  $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{n}$ .
- 2. The Killing form  $B$  of  $\mathfrak{m}$  vanishes on  $\mathfrak{n}$ .

*Proof of Lemma 3.12.* The first statement follows from (3.11.B).

The Killing form  $B$  of  $\mathfrak{m}$  is defined by

$$B(x, y) = \text{tr}|_{\mathfrak{m}} (ad x|_{\mathfrak{m}} \cdot ad y|_{\mathfrak{m}}).$$

Now the second statement follows from (3.11.B).  $\square$

According to Theorem 2.1.2 in [GG], conditions 1 and 2 for a finite-dimensional Lie algebra  $\mathfrak{m}$  imply that  $\mathfrak{m}$  is solvable. Thus,  $\mathfrak{m}$  is a solvable subalgebra of  $\mathfrak{g}$ . This, in particular, means that  $\mathfrak{m}$  is contained in a Borel subalgebra  $\mathfrak{b}$ . Since  $\mathfrak{h} \subset \mathfrak{b}$ , the Borel subalgebra  $\mathfrak{b}$  defines a polarization of roots  $\Delta = \Delta_+ \cup \Delta_-$  such that  $Y \subset \Delta_+$ . Theorem 3.11 is proved.  $\square$

*Proof of Theorem 3.10.* To prove the first statement of Theorem 3.10 it is enough to check that for any roots  $\alpha, \beta, \gamma \in \Delta$  such that  $\alpha + \beta + \gamma = 0$ , the functions  $\varphi_\alpha, \varphi_\beta, \varphi_\gamma$  satisfy equation (3.15). This could be easily done by direct verification.

Now we prove the second statement of Theorem 3.10. Let  $r : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  be a classical dynamical r-matrix with nonzero coupling constant  $\epsilon$ . According to Section 3.6, the r-matrix has the form

$$r(\lambda) = \sum_{i,j=1}^N C_{i,j}(\lambda) x_i \otimes x_j + \frac{\epsilon}{2} \Omega + \sum_{\alpha \in \Delta} \varphi_\alpha(\lambda) e_\alpha \otimes e_{-\alpha},$$

where  $\sum_{i,j} C_{i,j} dx_i \otimes dx_j$  is a closed meromorphic 2-form on  $\mathfrak{h}^*$ , and the functions  $\varphi_\alpha$  are scalar meromorphic functions such that  $\varphi_\alpha = \frac{\epsilon}{2} \cotanh(\frac{\epsilon}{2}((\alpha, \lambda) - \mu_\alpha))$  for a suitable constant  $\mu_\alpha$  or  $\varphi_\alpha = \pm \epsilon/2$ . Moreover, for any roots  $\alpha, \beta, \gamma \in \Delta$  such that  $\alpha + \beta + \gamma = 0$ , the functions  $\varphi_\alpha, \varphi_\beta, \varphi_\gamma$  satisfy equation (3.15).

Let  $Y \subset \Delta$  be the set of all roots  $\alpha$  such that  $\varphi_\alpha = \epsilon/2$ . Equation (3.15) and the unitarity condition easily imply that the set  $Y$  has properties 3.11.A - 3.11.B. By Theorem 3.11 there exists a polarization of roots  $\Delta = \Delta_+ \cup \Delta_-$  such that  $Y \subset \Delta_+$ .

Introduce two sets  $X$  and  $Z$  by

$$X = \Delta_+^s - \Delta_+^s \cap Y, \quad Z = \Delta_+ - Y.$$



**Lemma 3.13.**

$Z$  is the span of  $X$  in  $\Delta_+$ , i.e.  $Z = \mathbb{Z}_{\geq 0}[X] \cap \Delta_+$ .

*Proof.* If  $\alpha, \beta, \alpha + \beta \in \Delta_+$ ,  $\varphi_\alpha \neq \pm\epsilon/2$  and  $\varphi_\beta \neq \pm\epsilon/2$ , then equation (3.15) implies that  $\varphi_{\alpha+\beta} \neq \pm\epsilon/2$ . This statement implies the inclusion  $\mathbb{Z}_{\geq 0}[X] \subset Z$ .

If  $\alpha, \beta, \alpha + \beta \in \Delta_+$  and  $\varphi_\alpha = \epsilon/2$ , then equation (3.15) implies that  $\varphi_{\alpha+\beta} = \epsilon/2$  (since  $\varphi_\beta$  could not be equal to  $-\epsilon/2$ ). This statement implies the inclusion  $Z \subset \mathbb{Z}_{\geq 0}[X]$ . The Lemma is proved.  $\square$

For  $\alpha \in Z$  the functions  $\varphi_\alpha$  have the form  $\varphi_\alpha = \frac{\epsilon}{2} \cotanh(\frac{\epsilon}{2}((\alpha, \lambda) - \mu_\alpha))$  where  $\mu_\alpha$  are suitable numbers. Moreover, if  $\alpha, \beta, \alpha + \beta \in Z$ , then the corresponding constants  $\mu_\alpha, \mu_\beta, \mu_{\alpha+\beta}$  satisfy the equation  $\mu_\alpha + \mu_\beta = \mu_{\alpha+\beta}$ .

Let  $\mu \in \mathfrak{h}^*$  be any element in the Cartan subalgebra such that  $\mu_\alpha = (\alpha, \mu)$  for all  $\alpha \in X$ . Then for all  $\alpha \in Z$ , we have  $\mu_\alpha = (\alpha, \mu)$ .

Now we may conclude that the r-matrix  $r$  has the form indicated in Theorem 3.10 and is associated to the polarization  $\Delta = \Delta_+ \cup \Delta_-$ , the set  $X$  and the element  $\mu$  constructed above. Theorem 3.10 is proved.  $\square$

Now we show that each of the r-matrices indicated in Theorem 3.10 is a limiting case of the r-matrix (3.4). Namely, let  $\Delta = \Delta_+ \cup \Delta_-$  be a polarization,  $X \subset \Delta_+^s$  a subset of positive simple roots,  $\mu$  an element of  $\mathfrak{h}^*$ ,  $\epsilon$  a nonzero complex number. Set

$$\nu(t) = \mu + t \sum_{i \in X} \omega_i$$

where  $\omega_i$  are fundamental weights of  $\mathfrak{g}$ . Consider the r-matrix  $r_t(\lambda)$  defined by (3.4) with parameter  $\nu$  equal to  $\nu(t)$ . Then the limit of  $r_t(\lambda)$  as  $t$  tends to infinity is equal to the r-matrix of Theorem 3.10 (unlike the example at the end of Section 3.4).

**Example.** For any polarization, the constant matrix

$$(3.16) \quad r = \frac{1}{2} \sum_i x_i \otimes x_i + \sum_{\alpha \in \Delta_+} e_\alpha \otimes e_{-\alpha}$$

is a solution to the CDYB equation with coupling constant 1. In particular, it is a solution to the classical Yang-Baxter equation. This solution corresponds to  $X = \Delta_+^s$ .

**Remark.** Consider the r-matrix (3.4) with  $C_{i,j} = 0$  and  $\nu = 0$ ,

$$r(\lambda) = \frac{\epsilon}{2} \Omega + \sum_{\alpha \in \Delta} \frac{\epsilon}{2} \cotanh\left(\frac{\epsilon}{2}(\alpha, \lambda)\right) e_\alpha \otimes e_{-\alpha}.$$

Let  $\Gamma \subset \mathfrak{h}^*$  be an alcove of the Lie algebra  $\mathfrak{g}$  and  $\Delta = \Delta_+ \cup \Delta_-$  the corresponding polarization. If  $\lambda$  tends to infinity inside the alcove  $\Gamma$  in a generic direction, then the r-matrix  $r(\lambda)$  has a limit and this limit is given by (3.16). Thus the r-matrix  $r(\lambda)$  extrapolates different solutions of the classical Yang-Baxter equation of type (3.16), labeled by different polarizations.

### 3.8. Classical dynamical r-matrices associated with a pair of Lie algebras.

Let  $\mathfrak{g}$  be a simple Lie algebra,  $\mathfrak{h}$  its Cartan subalgebra,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

the root decomposition,  $(\cdot, \cdot)$  an invariant nondegenerate bilinear form on  $\mathfrak{g}$  and  $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$  the associated Casimir operator. Let  $\mathfrak{l}$  be a Lie subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{h}$ . Assume that  $\mathfrak{l}$  is reductive. This condition is equivalent to the condition that there is a subset  $\Delta(\mathfrak{l})_+$  of the set  $\Delta_+$  of positive roots such that

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

Let  $H \subset L \subset G$  be the corresponding complex Lie groups.

A meromorphic function

$$r : \mathfrak{l}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$$

is called a *classical dynamical r-matrix associated with the pair  $\mathfrak{l} \subset \mathfrak{g}$*  if it satisfies the following three conditions.

1. *The invariance condition.* The meromorphic function  $r$  is  $\text{Ad } L$  - invariant,  $r(x\lambda x^{-1}) = \text{Ad } x(r(\lambda))$  for any  $\lambda \in \mathfrak{l}^*$  and  $x \in L$ .
2. *The generalized unitarity,*

$$(3.17) \quad r^{12}(\lambda) + r^{21}(\lambda) = \epsilon \Omega$$

for some constant  $\epsilon \in \mathbb{C}$  and all  $\lambda$ .

3. *The classical dynamical Yang-Baxter equation, CDYBE,*

$$(3.18) \quad \text{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

The differential of the r-matrix is considered in (3.18) as a meromorphic function

$$dr : \mathfrak{l}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}, \quad \lambda \mapsto \sum_i y_i \otimes \frac{\partial r^{23}}{\partial y_i}(\lambda),$$

where  $\{y_i\}$  is any basis in  $\mathfrak{l}$ . As before we denote by  $\text{Alt}(dr)$  the following symmetrization of  $dr$ ,

$$\text{Alt}(dr) = \sum_i y_i^{(1)} \frac{\partial r^{23}}{\partial y_i} + \sum_i y_i^{(2)} \frac{\partial r^{31}}{\partial y_i} + \sum_i y_i^{(3)} \frac{\partial r^{12}}{\partial y_i}.$$

It turns out that the classification of the classical dynamical r-matrices associated with a pair  $\mathfrak{l} \subset \mathfrak{g}$  can be reduced to the classification of the the classical dynamical r-matrices associated with the pair  $\mathfrak{h} \subset \mathfrak{g}$ .

Namely, let us identify  $\mathfrak{l}^*$  with  $\mathfrak{h}^* \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})_+} (\mathfrak{g}_\alpha^* \oplus \mathfrak{g}_{-\alpha}^*)$ . Let  $r : \mathfrak{l}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  be a classical dynamical r-matrix associated with a pair  $\mathfrak{l} \subset \mathfrak{g}$ . First notice that if  $\lambda \in \mathfrak{l}^*$  is a semisimple element, then there exists  $x \in L$  such that  $x\lambda x^{-1} \in \mathfrak{h}^*$ . Since semisimple elements are dense and since the r-matrix satisfies the invariance condition, the function  $r$  is completely determined by its restriction to the dual to the Cartan subalgebra,  $r|_{\mathfrak{h}^*}$ .

For any  $\alpha \in \Delta(\mathfrak{l})_+$  fix basis elements  $e_\alpha \in \mathfrak{g}_\alpha$  and  $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$  which are dual with respect to the invariant bilinear form. Define a function  $\rho : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  by the formula

$$(3.19) \quad \rho(\lambda) = \sum_{\alpha \in \Delta(\mathfrak{l})_+} \frac{e_\alpha \otimes e_{-\alpha} - e_{-\alpha} \otimes e_\alpha}{(\alpha, \lambda)}$$

**Theorem 3.14.**

A function  $r : \mathfrak{l}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  satisfying the invariance condition is a classical dynamical  $r$ -matrix associated with the pair  $\mathfrak{l} \subset \mathfrak{g}$  if and only if the function

$$r|_{\mathfrak{h}^*} + \rho : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$$

is a classical dynamical  $r$ -matrix associated with the pair  $\mathfrak{h} \subset \mathfrak{g}$ .

Theorem 3.14 is proved in Section 3.9.

**3.9. Proof of Theorem 3.14.**

Let  $\{x_i\}$  be any basis in  $\mathfrak{h}$ . The elements  $\{x_i\}$ , and  $e_\alpha, e_{-\alpha}$ ,  $\alpha \in \Delta(\mathfrak{l})_+$ , form a basis in  $\mathfrak{l}$ . Then

$$(3.20) \quad dr(\lambda) = \sum_i x_i \otimes \frac{\partial r^{23}}{\partial x_i}(\lambda) + \sum_{\alpha \in \Delta(\mathfrak{l})_+} (e_\alpha \otimes \frac{\partial r^{23}}{\partial e_\alpha}(\lambda) + e_{-\alpha} \otimes \frac{\partial r^{23}}{\partial e_{-\alpha}}(\lambda))$$

for any  $\lambda \in \mathfrak{l}^*$ .

**Lemma 3.15.**

Let  $\lambda \in \mathfrak{h}^*$ . Then the second sum in (3.20) is equal to

$$[\rho^{12}(\lambda) + \rho^{13}(\lambda), r^{23}(\lambda)].$$

*Proof.* Let  $\{x_i^*\}$ , and  $e_\alpha^*, e_{-\alpha}^*$ ,  $\alpha \in \Delta(\mathfrak{l})_+$  be the basis in  $\mathfrak{l}^*$  dual to the basis  $\{x_i\}$ ,  $e_\alpha, e_{-\alpha}$ ,  $\alpha \in \Delta(\mathfrak{l})_+$ . The Lie algebra  $\mathfrak{l}$  acts on  $\mathfrak{l}^*$  by  $\text{ad}^*$ . Compute this action on elements of  $\mathfrak{h}^* \subset \mathfrak{l}^*$ . For  $\lambda \in \mathfrak{h}^*$  we have  $((\text{ad}^* e_\alpha) \lambda)(y) = -\lambda([e_\alpha, y])$ . Hence  $(\text{ad}^* e_\alpha) \lambda = -\lambda(h_\alpha) e_{-\alpha}^* = -(\alpha, \lambda) e_{-\alpha}^*$ . Similarly  $(\text{ad}^* e_{-\alpha}) \lambda = (\alpha, \lambda) e_\alpha^*$  and  $(\text{ad}^* x_j) \lambda = 0$ .

Now for any  $\lambda \in \mathfrak{h}^*$  we have

$$(3.21) \quad \frac{\partial r}{\partial e_\alpha}(\lambda) = \frac{1}{(\alpha, \lambda)} [e_{-\alpha} \otimes 1 + 1 \otimes e_{-\alpha}, r(\lambda)].$$

Indeed,

$$r(e^{te_{-\alpha}} \lambda e^{-te_{-\alpha}}) = r(\lambda) + t(\alpha, \lambda) \frac{\partial r}{\partial e_\alpha}(\lambda) + O(t^2),$$

and the invariance condition gives

$$\begin{aligned} r(e^{te_{-\alpha}} \lambda e^{-te_{-\alpha}}) &= (e^{te_{-\alpha}} \otimes e^{te_{-\alpha}}) r(\lambda) (e^{-te_{-\alpha}} \otimes e^{-te_{-\alpha}}) = \\ &= r(\lambda) + t[e_{-\alpha} \otimes 1 + 1 \otimes e_{-\alpha}, r(\lambda)] + O(t^2), \end{aligned}$$

which implies (3.21).

Similarly

$$(3.22) \quad \frac{\partial r}{\partial e_{-\alpha}}(\lambda) = \frac{-1}{(\alpha, \lambda)} [e_\alpha \otimes 1 + 1 \otimes e_\alpha, r(\lambda)].$$

Formulae (3.21) and (3.22) prove the Lemma.  $\square$

Denote the first sum in (3.20) by  $d_{\mathfrak{h}}r$ , then for  $\lambda \in \mathfrak{h}^*$  we have

$$dr(\lambda) = d_{\mathfrak{h}}r(\lambda) + [\rho^{12}(\lambda) + \rho^{13}(\lambda), r^{23}(\lambda)].$$

Now we finish the proof of Theorem 3.14. Let  $r : \mathfrak{l}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  be a function satisfying the invariance condition. Restrict this function to  $\mathfrak{h}^*$ . The invariance condition implies that the restriction satisfies the zero weight condition (3.1). The restriction also satisfies the generalized unitarity condition (3.17). Introduce a function  $\tilde{r} : \mathfrak{h}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  by

$$r|_{\mathfrak{h}^*}(\lambda) = \tilde{r}(\lambda) - \rho(\lambda).$$

Then the new function  $\tilde{r}(\lambda)$  satisfies the zero weight condition (3.1) and the generalized unitarity condition (3.17).

**Lemma 3.16.**

*The function  $r$  satisfies the CDYB equation (3.18) on  $\mathfrak{h}^*$  if and only if the function  $\tilde{r}$  satisfies the CDYB equation (3.3).*

Lemma 3.16 implies Theorem 3.14.

*Proof of Lemma 3.16.* The restriction of the CDYB equation on  $\mathfrak{h}^*$  takes the form

$$(3.23) \quad \begin{aligned} & \text{Alt}(d_{\mathfrak{h}}\tilde{r}) - \text{Alt}(d_{\mathfrak{h}}\rho) - [\rho^{12} + \rho^{13}, \rho^{23}] - [\rho^{21} + \rho^{23}, \rho^{31}] \\ & - [\rho^{31} + \rho^{32}, \rho^{12}] + [\rho^{12}, \rho^{13}] + [\rho^{12}, \rho^{23}] + [\rho^{13}, \rho^{23}] \\ & - [\rho^{12} + \rho^{13}, \tilde{r}^{23}] - [\rho^{21} + \rho^{23}, \tilde{r}^{31}] - [\rho^{31} + \rho^{32}, \tilde{r}^{12}] \\ & - [\tilde{r}^{12}, \rho^{13}] - [\rho^{12}, \tilde{r}^{13}] - [\tilde{r}^{12}, \rho^{23}] - [\rho^{12}, \tilde{r}^{23}] - [\tilde{r}^{13}, \rho^{23}] \\ & - [\rho^{13}, \tilde{r}^{23}] + [\tilde{r}^{12}, \tilde{r}^{13}] + [\tilde{r}^{12}, \tilde{r}^{23}] + [\tilde{r}^{13}, \tilde{r}^{23}]. \end{aligned}$$

Using the identity  $\rho^{12}(\lambda) + \rho^{21}(\lambda) = 0$  we conclude that the terms containing only the function  $\rho$  take the form of the CDYB equation (3.3). By Theorem 3.2 we know that the function  $\rho$  satisfies equation (3.3), and hence the  $\rho$ -terms in (3.23) are canceled out.

Using in addition the generalized unitarity condition for  $\tilde{r}$  we see that the  $[\rho, \tilde{r}]$ -terms in (3.23) are canceled out too. The remaining part of (3.23) is the CDYB equation (3.3) for the function  $\tilde{r}$ .

Lemma 3.16 and Theorem 3.14 are proved.  $\square$

#### 4. CLASSIFICATION OF CLASSICAL DYNAMICAL r-MATRICES WITH SPECTRAL PARAMETER

##### 4.1. Classical dynamical r-matrices with spectral parameter.

Let  $\mathfrak{g}$  be a simple Lie algebra,  $\mathfrak{h}$  its Cartan subalgebra,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}$$

the root decomposition,  $(\cdot, \cdot)$  an invariant nondegenerate bilinear form on  $\mathfrak{g}$  and  $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$  the associated Casimir operator. For any positive root  $\alpha \in \Delta(\mathfrak{h})$  fix basis elements  $e_{\alpha} \in \mathfrak{g}_{\alpha}$  and  $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$  which are dual with respect to the bilinear

form. Fix a basis in the Cartan subalgebra,  $\{x_i\}$ , orthonormal with respect to the bilinear form. Let  $E$  be a neighborhood of 0 in  $\mathbb{C}$ ,  $0 \in E \subset \mathbb{C}$ .

A meromorphic function

$$r : \mathfrak{h}^* \times E \rightarrow \mathfrak{g} \otimes \mathfrak{g}$$

is called a *classical dynamical r-matrix with spectral parameter and associated with the pair  $\mathfrak{h} \subset \mathfrak{g}$*  if it satisfies the following four conditions.

1. *The zero weight condition,*

$$(4.1) \quad [h \otimes 1 + 1 \otimes h, r(\lambda, z)] = 0$$

for all  $\lambda \in \mathfrak{h}^*$ ,  $z \in E$  and  $h \in \mathfrak{h}$ .

2. *The generalized unitarity,*

$$(4.2) \quad r^{12}(\lambda, z) + r^{21}(\lambda, -z) = 0.$$

for all  $\lambda \in \mathfrak{h}^*$  and  $z \in E$ .

3. *The residue condition.*

$$(4.3) \quad \text{Res}_{z=0} r(\lambda, z) = \epsilon \Omega.$$

for some constant  $\epsilon \in \mathbb{C}$ .

4. *The classical dynamical Yang-Baxter equation, CDYBE,*

$$(4.4) \quad \text{Alt}(d_{\mathfrak{h}}r) + [r^{12}(\lambda, z_{1,2}), r^{13}(\lambda, z_{1,3})] + \\ [r^{12}(\lambda, z_{1,2}), r^{23}(\lambda, z_{2,3})] + [r^{13}(\lambda, z_{1,3}), r^{23}(\lambda, z_{2,3})] = 0.$$

where  $z_{i,j} = z_i - z_j$ .

In (4.4) the differential of the r-matrix is considered with respect to the  $\mathfrak{h}$ -variables,

$$d_{\mathfrak{h}}r : \mathfrak{h}^* \times E \rightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}, \quad (\lambda, z) \mapsto \sum_i x_i \otimes \frac{\partial r^{23}}{\partial x_i}(\lambda, z).$$

In (4.4) we denote by  $\text{Alt}(d_{\mathfrak{h}}r)$  the following symmetrization of  $d_{\mathfrak{h}}r$ ,

$$\text{Alt}(d_{\mathfrak{h}}r) = \sum_i x_i^{(1)} \frac{\partial r^{23}}{\partial x_i}(\lambda, z_{2,3}) + \sum_i x_i^{(2)} \frac{\partial r^{31}}{\partial x_i}(\lambda, z_{3,1}) + \sum_i x_i^{(3)} \frac{\partial r^{12}}{\partial x_i}(\lambda, z_{1,2}).$$

The variable  $z$  is called *the spectral parameter*. The number  $\epsilon$  in (4.3) is called *the coupling constant*.

We classify the germs of classical dynamical r-matrices with spectral parameter at the subset  $\mathfrak{h}^* \times 0 \subset \mathfrak{h}^* \times \mathbb{C}$ . We assume that an r-matrix has a Laurent power series expansion of the form  $r(\lambda, z) = \sum_m r^m(\lambda) z^m$  where  $r^m(\lambda)$  are meromorphic functions on  $\mathfrak{h}^*$ . We assume that the Laurent expansion is convergent to a meromorphic function of  $\lambda$  and  $z$  in a punctured neighborhood of  $\mathfrak{h}^* \times 0$  in  $\mathfrak{h}^* \times \mathbb{C}$ . Any function  $r(\lambda, z)$  with these properties will be called a function with Laurent expansion.

## 4.2. Gauge transformations.

In this subsection we introduce four transformations of maps  $r : \mathfrak{h}^* \times \mathbb{C} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  called *the gauge transformations*. We assume that the map satisfies the zero weight condition and the generalized unitarity condition and, therefore, has the form

$$(4.5) \quad r(\lambda, z) = \sum_{i,j=1}^N S_{i,j}(\lambda, z) x_i \otimes x_j + \sum_{\alpha \in \Delta} \varphi_\alpha(\lambda, z) e_\alpha \otimes e_{-\alpha}$$

where  $\varphi_\alpha, S_{i,j}$  are suitable scalar meromorphic functions such that  $\varphi_{-\alpha}(\lambda, -z) = -\varphi_\alpha(\lambda, z)$ , and  $S_{i,j}(\lambda, -z) = -S_{j,i}(\lambda, z)$ .

1. Let  $C = \sum_{i,j} C_{i,j}(\lambda) dx_i \otimes dx_j$  be a closed meromorphic 2-form. Set

$$r(\lambda, z) \mapsto r(\lambda, z) + \sum_{i,j=1}^N C_{i,j}(\lambda) x_i \otimes x_j.$$

2. For a vector  $v \in \mathfrak{h}^*$  and a function  $f$  on  $\mathfrak{h}^*$ , we denote  $L_v f$  a new function on  $\mathfrak{h}^*$  which is the derivative of  $f$  along the constant vector field defined by  $v$ .

For a holomorphic function  $\psi : \mathfrak{h}^* \rightarrow \mathbb{C}$ , set

$$r(\lambda, z) \mapsto \sum_{i,j=1}^N \left( S_{i,j}(\lambda, z) + z \frac{\partial^2 \psi}{\partial x_i \partial x_j}(\lambda) \right) x_i \otimes x_j + \sum_{\alpha \in \Delta} \varphi_\alpha(\lambda, z) e^{z L_\alpha \psi(\lambda)} e_\alpha \otimes e_{-\alpha}.$$

3. For  $\nu \in \mathfrak{h}^*$ , set

$$r(\lambda, z) \mapsto \sum_{i,j=1}^N S_{i,j}(\lambda - \nu, z) x_i \otimes x_j + \sum_{\alpha \in \Delta} \varphi_\alpha(\lambda - \nu, z) e_\alpha \otimes e_{-\alpha}.$$

4. For nonzero complex number  $a, b$ , set

$$r(\lambda, z) \mapsto a r(a\lambda, bz).$$

Notice that the first three transformations do not change the residue of  $r(\lambda, z)$  at  $z = 0$  and the last transformation multiplies it by  $a/b$ .

**Theorem 4.1.** *Any gauge transformation transforms an  $r$ -matrix with spectral parameter to an  $r$ -matrix with spectral parameter.*

Theorem 4.1 is proved in Section 4.5.

Two maps  $r(\lambda, z)$  and  $r'(\lambda, z)$  will be called *equivalent* if one of them could be transformed into another by a sequence of gauge transformations.

## 4.3. Elliptic, trigonometric and rational $r$ -matrices with spectral parameter.

In this section we give examples of  $r$ -matrices with spectral parameter, and formulate a theorem that any  $r$ -matrix with nonzero coupling constant is equivalent to one of the examples.

In order to describe r-matrices we use theta functions. Let

$$\theta_1(z, \tau) = - \sum_{j=-\infty}^{\infty} e^{\pi i(j+\frac{1}{2})^2 \tau + 2\pi i(j+\frac{1}{2})(z+\frac{1}{2})}$$

be the Jacobi theta function. Let  $\tau$  be a nonzero complex number such that  $\text{Im } \tau > 0$ . Following [FW] introduce the functions

$$(4.6) \quad \sigma_w(z, \tau) = \frac{\theta_1(w - z, \tau) \theta_1'(0, \tau)}{\theta_1(w, \tau) \theta_1(z, \tau)}, \quad \rho(z, \tau) = \frac{\theta_1'(z, \tau)}{\theta_1(z, \tau)},$$

where  $'$  means the derivative with respect to the first argument. Notice that  $\sigma_w(z, \tau) = z^{-1} + O(1)$ ,  $\rho(z, \tau) = z^{-1} + O(z)$  as  $z \rightarrow 0$  and  $\sigma_{-w}(-z, \tau) = -\sigma_w(z, \tau)$ ,  $\rho(-z, \tau) = -\rho(z, \tau)$ .

### Example of an elliptic r-matrix.

$$(4.7) \quad r(\lambda, z, \tau) = \rho(z, \tau) \sum_{i=1}^N x_i \otimes x_i + \sum_{\alpha \in \Delta} \sigma_{-(\alpha, \lambda)}(z, \tau) e_{\alpha} \otimes e_{-\alpha}.$$

For every  $\tau \in \mathbb{C}$ ,  $\text{Im } \tau > 0$ , the function  $r(\lambda, z, \tau)$  is a classical dynamical r-matrix with spectral parameter  $z$  and coupling constant  $\epsilon = 1$  [FW].

### Examples of trigonometric r-matrices.

Let  $\Delta = \Delta_+ \cup \Delta_-$  be a polarization of the set of roots. Fix a subset  $X \subset \Delta_+^s$  of the set of simple positive roots. For any root  $\alpha$  introduce a meromorphic function  $\varphi_{\alpha} : \mathfrak{h}^* \rightarrow \mathbb{C}$  by the following rule. If a root  $\alpha$  is a linear combination of simple roots from  $X$ , then we set

$$\varphi_{\alpha}(\lambda, z) = \frac{\sin((\alpha, \lambda) + z)}{\sin(\alpha, \lambda) \sin z},$$

otherwise we set

$$\varphi_{\alpha}(\lambda, z) = \frac{e^{-iz}}{\sin z}, \quad \text{for } \alpha \in \Delta_+, \quad \varphi_{\alpha}(\lambda, z) = \frac{e^{iz}}{\sin z}, \quad \text{for } \alpha \in \Delta_-.$$

We introduce a trigonometric r-matrix by

$$(4.8) \quad r(\lambda, z) = \cotan z \sum_{i=1}^N x_i \otimes x_i + \sum_{\alpha \in \Delta} \varphi_{\alpha}(\lambda, z) e_{\alpha} \otimes e_{-\alpha}$$

where  $\cotan z = \cos z / \sin z$ .

### Examples of rational r-matrices.

For a subset  $X \subset \Delta$  of the set of roots closed with respect to addition and multiplication by  $-1$ , we introduce a rational r-matrix by

$$(4.9) \quad r(\lambda, z) = \frac{\Omega}{z} + \sum_{\alpha \in X} \frac{1}{(\alpha, \lambda)} e_{\alpha} \otimes e_{-\alpha}.$$

**Theorem 4.2.**

1. Each of the  $r$ -matrices (4.7) - (4.9) described in this section is a classical dynamical  $r$ -matrix with coupling constant 1.
2. The germ at  $\mathfrak{h}^* \times 0 \subset \mathfrak{h}^* \times \mathbb{C}$  of any classical dynamical  $r$ -matrix with spectral parameter and a nonzero coupling constant is equivalent to one of the  $r$ -matrices (4.7)- (4.9).

**Corollary.** Any such a germ extends to a meromorphic function on  $\mathfrak{h}^* \times \mathbb{C}$ .

Theorem 4.2 is proved in Sections 4.4 and 4.5.

**4.4. Proof of part 1 of Theorem 4.2.** According to [FW], the elliptic  $r$ -matrix (4.7) is a classical dynamical  $r$ -matrix with spectral parameter.

Taking the limit of the  $r$ -matrix (4.7) when  $\tau$  tends to  $+i\infty$ , we conclude that for any fixed element  $\nu \in \mathfrak{h}$  the  $r$ -matrix

$$r_\nu(\lambda, z) = \cotan z \sum_{i=1}^N x_i \otimes x_i + \sum_{\alpha \in \Delta} \frac{\sin((\alpha, \lambda - \nu) + z)}{\sin(\alpha, \lambda - \nu) \sin z} e_\alpha \otimes e_{-\alpha}$$

is a classical dynamical  $r$ -matrix with spectral parameter. If  $\nu = 0$ , then this  $r$ -matrix has form (4.8) with  $X = \Delta_+^s$ . To show that any matrix of the form of (4.8) is a classical dynamical  $r$ -matrix with spectral parameter it is enough to apply to  $r_\nu$  the limiting procedure with respect to  $\nu$ , cf. the end of Section 3.7.

**Lemma 4.3.** If  $r_0(\lambda)$  is a classical dynamical  $r$ -matrix without spectral parameter and with zero coupling constant, then

$$r(\lambda, z) = \frac{\Omega}{z} + r_0(\lambda)$$

is a classical dynamical  $r$ -matrix with spectral parameter.

Proof by direct verification.  $\square$

Lemma 4.3 and Theorem 3.2 show that any  $r$ -matrix (4.9) is a classical dynamical  $r$ -matrix with spectral parameter.

**4.5. Proof of part 2 of Theorem 4.1 and Theorem 4.2.**

Let  $r : \mathfrak{h}^* \times \mathbb{C} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  be a germ at  $\mathfrak{h}^* \times 0 \subset \mathfrak{h}^* \times \mathbb{C}$  of a classical dynamical  $r$ -matrix with spectral parameter. It follows from the zero weight condition that the  $r$ -matrix can be written in the form of (4.5).

The CDYB equation (4.4) is an equation in  $\mathfrak{g}^{\otimes 3}$ . Its left hand side is invariant with respect to even permutations of factors and simultaneous permutations of variables  $z_1, z_2, z_3$ . In order to solve the CDYB equation it suffices to solve its  $\mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h}-$ ,  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}-$  and  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_\gamma-$ parts, where  $\alpha, \beta, \gamma \in \Delta$  and in the last case  $\alpha + \beta + \gamma = 0$ .

**4.5.1. The  $\mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h}$ -part of the CDYB equation.**

First we analyse the  $\mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h}$ -part of the CDYB equation (4.4) which has the form

$$(4.10) \quad \sum_i x_i^{(1)} \frac{\partial S^{23}}{\partial x_i}(\lambda, z_2 - z_3) + \sum_i x_i^{(2)} \frac{\partial S^{31}}{\partial x_i}(\lambda, z_3 - z_1) + \sum_i x_i^{(3)} \frac{\partial S^{12}}{\partial x_i}(\lambda, z_1 - z_2) = 0.$$



**Lemma 4.4.**

The sum  $\sum_i x_i^{(1)} \frac{\partial S^{23}}{\partial x_i}(\lambda, z)$  is a linear function of  $z$ .

*Proof.* Differentiating (4.10) with respect to  $z_1$  and  $z_2$  we conclude that the second derivative  $\partial^2/\partial z_1 \partial z_2$  of the third sum in (4.10) is equal to zero. This implies the Lemma.  $\square$

**Corollary 4.5.**

Consider the  $\mathfrak{h} \otimes \mathfrak{h}$ -valued function  $S(\lambda, z) = \sum_{i,j} S_{i,j}(\lambda, z) x_i \otimes x_j$ . Let  $S(\lambda, z) = \sum_n S^n(\lambda) z^n$  be its Laurent expansion. Then

$$\frac{\partial S^n}{\partial x_i}(\lambda) = 0$$

for all  $\lambda$ ,  $i$ , and  $n$ ,  $n \neq 0, 1$ .

Each of the three sums in (4.10) is a linear function of  $z_1, z_2, z_3$ . Hence equation (4.10) splits into four independent equations corresponding to the coefficients of  $z_1, z_2, z_3$  and the constant coefficient. The constant coefficient part has the form

$$\frac{\partial S_{j,k}^0}{\partial x_i} + \frac{\partial S_{k,i}^0}{\partial x_j} + \frac{\partial S_{i,j}^0}{\partial x_k} = 0$$

and is equivalent to the fact that  $\sum S_{i,j}^0(\lambda) dx_i \otimes dx_j$  is a closed differential form.

**Lemma 4.6.**

There is a multivalued meromorphic function  $\psi : \mathfrak{h}^* \rightarrow \mathbb{C}$  with univalued meromorphic second derivatives such that

$$\frac{\partial^2 \psi}{\partial x_i \partial x_j}(\lambda) = S_{i,j}^1(\lambda)$$

for all  $i, j, \lambda$ .

*Proof.* The  $z_1, z_2, z_3$ -parts of equation (4.10) together with the unitarity condition  $S_{i,j}^1(\lambda) = S_{j,i}^1(\lambda)$  have the form

$$\frac{\partial S_{i,j}^1}{\partial x_k}(\lambda) = \frac{\partial S_{k,j}^1}{\partial x_i}(\lambda)$$

for all  $\lambda, i, j, k$ . These equations imply that there exist functions  $\varphi_j$  such that  $S_{i,j}^1 = \partial \varphi_j / \partial x_i$  and moreover,  $\partial \varphi_j / \partial x_i = \partial \varphi_i / \partial x_j$ . Hence, there exists a function  $\psi$  with the properties indicated in the Lemma.  $\square$

**Remark.** Later we will show that the function  $\psi$  is in fact holomorphic in  $\mathfrak{h}^*$ .

**Corollary 4.7.**

Let  $s : \mathbb{C} \rightarrow \mathfrak{h} \otimes \mathfrak{h}$  be a germ at  $\mathfrak{h}^* \times 0 \subset \mathfrak{h}^* \times \mathbb{C}$  of a meromorphic function with Laurent expansion. Assume that

$$s(z) + s^{21}(-z) = 0$$

for all  $z$ , and assume that the Laurent expansion of  $s$  does not contain the terms of degree 0, 1,  $s(z) = \sum_{m \neq 0,1} s^m z^m$ . Let  $r : \mathfrak{h}^* \times \mathbb{C} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  be a germ at  $\mathfrak{h}^* \times 0 \subset \mathfrak{h}^* \times \mathbb{C}$  of a function of the form

(4.11)

$$r(\lambda, z) = s(z) + \sum_{i,j=1}^N C_{i,j}(\lambda) x_i \otimes x_j + z \sum_{i,j=1}^N \frac{\partial^2 \psi}{\partial x_i \partial x_j}(\lambda) x_i \otimes x_j + \sum_{\alpha \in \Delta} \varphi_\alpha(\lambda, z) e_\alpha \otimes e_{-\alpha}$$

where  $\sum C_{i,j} dx_i \otimes dx_j$  is a closed meromorphic form on  $\mathfrak{h}^*$ ,  $\psi$  is a multivalued meromorphic function with univalued meromorphic second derivatives, and the functions  $\varphi_\alpha$  are such that  $\varphi_{-\alpha}(\lambda, -z) = -\varphi_\alpha(\lambda, z)$ . Then the function  $r$  satisfies the zero weight condition (4.1), the unitarity condition (4.2) and the  $\mathfrak{h} \otimes \mathfrak{h} \otimes \mathfrak{h}$ -part of the CDYB equation (4.4). Moreover, any classical dynamical  $r$ -matrix with spectral parameter has this form.

#### 4.5.2. The $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ -part of the CDYB equation.

Now we analyze the  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ -part of the CDYB equation. This part has the form

$$\begin{aligned} & \sum_i \frac{\partial \varphi_\alpha}{\partial x_i} x_i \otimes e_\alpha \otimes e_{-\alpha} + \varphi_{-\alpha}(\lambda, z_{1,2}) \varphi_\alpha(\lambda, z_{1,3}) [e_{-\alpha}, e_\alpha] \otimes e_\alpha \otimes e_{-\alpha} + \\ & \sum_{i,j} \varphi_\alpha(\lambda, z_{2,3}) (s_{i,j}(z_{1,2}) + C_{i,j}(\lambda) + z_{1,2} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(\lambda)) x_i \otimes [x_j, e_\alpha] \otimes e_{-\alpha} + \\ & \sum_{i,j} \varphi_\alpha(\lambda, z_{2,3}) (s_{i,j}(z_{1,3}) + C_{i,j}(\lambda) + z_{1,3} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(\lambda)) x_i \otimes e_\alpha \otimes [x_j, e_{-\alpha}] = 0. \end{aligned}$$

This equation can be written as

$$(4.12) \quad \begin{aligned} & -\varphi_{-\alpha}(\lambda, z_{1,2}) \varphi_\alpha(\lambda, z_{1,3}) h_\alpha \otimes e_\alpha \otimes e_{-\alpha} + \\ & \sum_i \left\{ \left( \frac{\partial \varphi_\alpha}{\partial x_i} + \varphi_\alpha(\lambda, z_{2,3}) \sum_j [s_{i,j}(z_{1,2}) - s_{i,j}(z_{1,3}) + z_{3,2} \alpha(x_j) \frac{\partial^2 \psi}{\partial x_i \partial x_j}(\lambda)] \right) x_i \otimes e_\alpha \otimes e_{-\alpha} \right\} = 0. \end{aligned}$$

We interpret this equation as an equation of differential 1-forms on  $\mathfrak{h}^*$  identifying linear functions with their differentials and ignoring the factor  $e_\alpha \otimes e_{-\alpha}$ . Then the first term in (4.12) can be written as  $\varphi_\alpha(\lambda, z_{2,1}) \varphi_\alpha(\lambda, z_{1,3}) dh_\alpha$ . The second has the form  $d\varphi_\alpha(\lambda, z_{2,3})$  where the differential is with respect to  $\lambda$ . For a fixed  $z$  consider  $\sum s_{i,j}(z) x_i \otimes x_j$  as a bilinear form on  $\mathfrak{h}^*$ ,

$$s(z)\{\lambda, \mu\} = \sum s_{i,j}(z) \langle x_i, \lambda \rangle \langle x_j, \mu \rangle.$$

If the second argument of this form is equal to  $\alpha$ , then we get a linear function on  $\mathfrak{h}^*$ ,  $\sum s_{i,j}(z) \langle x_j, \alpha \rangle x_i$ . Hence the third and the forth terms in (4.12) have the form

$\varphi_\alpha(\lambda, z_{2,3}) (ds(z_{1,2})\{\lambda, \alpha\} - ds(z_{1,3})\{\lambda, \alpha\})$  where the differentials are with respect to  $\lambda$ . Finally the last term in (4.12) could be written as  $z_{2,3}\varphi_\alpha(\lambda, z_{2,3}) L_\alpha d\psi(\lambda)$  where  $L_\alpha d\psi(\lambda)$  is the Lie derivative of the differential with respect to the constant vector field defined by  $\alpha$ . Now the  $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ -part of the CDYB equation takes the form

$$d\varphi_\alpha(\lambda, z_{2,3}) + \varphi_\alpha(\lambda, z_{2,1}) \varphi_\alpha(\lambda, z_{1,3}) dh_\alpha + \varphi_\alpha(\lambda, z_{2,3}) (ds(z_{1,2})\{\lambda, \alpha\} - ds(z_{1,3})\{\lambda, \alpha\} + z_{2,3} L_\alpha d\psi(\lambda)) = 0.$$

Setting  $u = z_{2,1}$  and  $v = z_{1,3}$  we get

$$\frac{d\varphi_\alpha(\lambda, u+v)}{\varphi_\alpha(\lambda, u+v)} + \frac{\varphi_\alpha(\lambda, u) \varphi_\alpha(\lambda, v)}{\varphi_\alpha(\lambda, u+v)} dh_\alpha + ds(-u)\{\lambda, \alpha\} - ds(v)\{\lambda, \alpha\} - (u+v) L_\alpha d\psi(\lambda) = 0$$

where the differentials are with respect to  $\lambda$ .

Make a change of variables,  $\varphi_\alpha(\lambda, z) = \Phi(\lambda, z) e^{zL_\alpha\psi(\lambda)}$ . Then

$$\frac{d\Phi(\lambda, u+v)}{\Phi(\lambda, u+v)} + \frac{\Phi(\lambda, u) \Phi(\lambda, v)}{\Phi(\lambda, u+v)} dh_\alpha + ds(-u)\{\lambda, \alpha\} - ds(v)\{\lambda, \alpha\} = 0.$$

Taking the differential of both sides we see that the second ratio depends only on  $h_\alpha$ . Hence the function  $\Phi$  has the form

$$\Phi(\lambda, z) = \mu(h_\alpha(\lambda), z) e^{z\nu(\lambda)}$$

for suitable new functions  $\mu(h_\alpha(\lambda), z)$  and  $\nu(\lambda)$ . Now the equation takes the form

$$(4.13) \quad \frac{\partial \mu(h_\alpha, u+v)/\partial h_\alpha}{\mu(h_\alpha, u+v)} dh_\alpha + (u+v) d\nu(\lambda) + \frac{\mu(h_\alpha, u) \mu(h_\alpha, v)}{\mu(h_\alpha, u+v)} dh_\alpha + ds(-u)\{\lambda, \alpha\} - ds(v)\{\lambda, \alpha\} = 0.$$

Consider a coordinate system on  $\mathfrak{h}^*$ ,  $y_1, \dots, y_N \in \mathfrak{h}$ , such that  $y_1 = h_\alpha$  and  $\langle \alpha, y_i \rangle = 0$  for  $i > 1$ . Then (4.13) gives the equations

$$(4.14) \quad \frac{\partial \mu(h_\alpha, u+v)/\partial h_\alpha}{\mu(h_\alpha, u+v)} + (u+v) \frac{\partial \nu}{\partial h_\alpha}(\lambda) + \frac{\mu(h_\alpha, u) \mu(h_\alpha, v)}{\mu(h_\alpha, u+v)} + \frac{s(-u)\{\alpha, \alpha\} - s(v)\{\alpha, \alpha\}}{(\alpha, \alpha)} = 0,$$

$$(4.15) \quad (u+v) \frac{\partial \nu}{\partial y_i}(\lambda) + s(-u)\left\{\frac{\partial}{\partial y_i}, \alpha\right\} - s(v)\left\{\frac{\partial}{\partial y_i}, \alpha\right\} = 0, \quad i = 2, \dots, N.$$

(Here  $\{\frac{\partial}{\partial y_i}\}$  is the basis of  $\mathfrak{h}^*$  dual to the basis  $\{y_i\}$  of  $\mathfrak{h}$ .) Equations (4.15) imply that  $\partial \nu / \partial y_i = 0$  since the Laurent expansion of the function  $s(z)$  does not have

the first order term. So we can assume that  $\nu = 0$  and, therefore,  $\varphi_\alpha(\lambda, z) = \mu(h_\alpha(\lambda), z) e^{zL_\alpha\psi(\lambda)}$ .

Now equations (4.15) imply that  $s(z)\{\frac{\partial}{\partial y_i}, \alpha\}$  does not depend on  $z$ . But the Laurent expansion of this function does not contain the terms of degree zero and one. So the function  $s(z)\{\frac{\partial}{\partial y_i}, \alpha\}$  is identically equal to zero for  $i \geq 2$ .

The fact that  $s(z)\{\frac{\partial}{\partial y_i}, \alpha\}$  is identically equal to zero for  $i \geq 2$  easily implies that

$$s(z) = t(z) \sum_{i=1}^N x_i \otimes x_i$$

where  $t(z)$  is a scalar function. The function  $t(z)$  is a nonzero function since the coupling constant is not zero. We have  $t(-z) = -t(z)$ , and the Laurent expansion of  $t$  does not have the terms of degree zero and one. Then (4.14) takes the form

$$(4.16) \quad \frac{\partial \mu(h_\alpha, u+v)}{\partial h_\alpha} = (t(u) + t(v))\mu(h_\alpha, u+v) - \mu(h_\alpha, u)\mu(h_\alpha, v).$$

#### 4.5.3. Proof of Theorem 4.1.

Before proceeding with analysis of equation (4.16) let us write the  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_\gamma$ -part of the CDYB equation,

$$(4.17) \quad \varphi_\alpha(\lambda, z_{13})\varphi_\beta(\lambda, z_{23}) + \varphi_\beta(\lambda, z_{21})\varphi_\gamma(\lambda, z_{31}) + \varphi_\alpha(\lambda, z_{12})\varphi_\gamma(\lambda, z_{32}) = 0.$$

It is easy to see that equations (4.10), (4.16), (4.17) are invariant with respect to the gauge transformations of Section 4.2. This proves Theorem 4.1.

#### 4.5.4. The $\mathfrak{h} \otimes \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ -part of the CDYB equation: classification of solutions.

Now we will find all solutions of equation (4.16).

#### Lemma 4.8.

*The function  $\mu(x, z)$  has at most a simple pole at  $z = 0$ .*

*Proof.* Applying the operator  $\partial/\partial u - \partial/\partial v$  to both sides of (4.16) we have

$$(t'(u) - t'(v))\mu(x, u+v) - \mu'(x, u)\mu(x, v) + \mu(x, u)\mu'(x, v) = 0.$$

Set  $v = -u + \delta$ . Then using the fact that  $t'(-z) = t'(z)$  we get

$$(t'(u) - t'(u - \delta))\mu(x, \delta) - \mu'(x, u)\mu(x, -u + \delta) + \mu(x, u)\mu'(x, -u + \delta) = 0.$$

The function  $t(z)$  could not be a linear function. Hence the first factor is of order  $\delta$ . The second and the third terms are regular at generic values of  $u$ . This shows the Lemma.  $\square$

The Lemma easily implies that the function  $t(z)$  has also at most a simple pole at  $z = 0$ . Thus,

$$\mu(x, z) = \sum_{n=-1}^{\infty} \mu_n(x)z^n, \quad t(z) = \sum_{n=-1, n \text{ odd}}^{\infty} t_n z^n.$$

Notice that from the residue condition (4.3) we know that  $t_{-1} = \epsilon(\alpha, \alpha)$  where  $\epsilon$  is the coupling constant. Substitute the expansions into (4.16) and multiply both sides by  $uv(u+v)$ ,

$$(4.18) \quad \sum_{n=-1}^{\infty} \mu'_n(x)(u+v)^{n+1}uv - \sum_{n,m=-1}^{\infty} t_n \mu_m(x)(u+v)^{m+1}(u^n + v^n)uv + \sum_{n,m=-1}^{\infty} \mu_n(x) \mu_m(x) u^{n+1} v^{m+1} (u+v) = 0.$$

This is an equality of power series in  $u$  and  $v$ . Equating the homogeneous parts we get a sequence of equations for the numbers  $\{t_n\}$  and the functions  $\{\mu_n(x)\}$ .

We write the first equations. The equation of degree 1 has the form

$$(4.19) \quad \mu_{-1}(x) = t_{-1}.$$

The equation of degree 2 has the form  $\mu'_{-1}(x) = 0$  and follows from (4.19). The equations of degree 3 and 4 have the form

$$\begin{aligned} \mu'_0 &= 2t_{-1}\mu_1 - \mu_0^2 + t_{-1}t_1, \\ \mu'_1 &= 3t_{-1}\mu_2 + t_1\mu_0 - \mu_0\mu_1. \end{aligned}$$

The equation of degree 5 gives two scalar equations

$$\begin{aligned} \mu'_2 &= 4t_{-1}\mu_3 + t_1\mu_1 + t_3t_{-1} - \mu_0\mu_2, \\ 2\mu'_2 &= 6t_{-1}\mu_3 + 2t_1\mu_1 - t_3t_{-1} - \mu_1^2, \end{aligned}$$

which imply

$$\begin{aligned} \mu'_2 &= t_1\mu_1 - 5t_{-1}t_3 + 3\mu_0\mu_2 - 2\mu_1^2, \\ 2\mu_3t_{-1} &= 2\mu_0\mu_2 - 3t_{-1}t_3 - \mu_1^2. \end{aligned}$$

Thus we have

$$(4.20) \quad \begin{aligned} \mu'_0 &= t_{-1}t_1 + 2t_{-1}\mu_1 - \mu_0^2, \\ \mu'_1 &= 3t_{-1}\mu_2 + t_1\mu_0 - \mu_0\mu_1, \\ \mu'_2 &= t_1\mu_1 - 5t_{-1}t_3 + 3\mu_0\mu_2 - 2\mu_1^2. \end{aligned}$$

**Lemma 4.9.**

*Let  $t_{-1} \neq 0$  and  $n \geq 3$ . Then the degree  $n+3$  equation determine  $\mu_{n+1}, t_{n+1}$  uniquely in terms of  $\mu_m, t_m$  with smaller  $m$ .*

*Proof.* The degree  $n+3$  equation contains only  $\mu_m, t_m$  with  $m \leq n+1$  and has the form

$$\begin{aligned} -t_{-1}\mu_{n+1}(u+v)^{n+2}\left(\frac{1}{u} + \frac{1}{v}\right)uv - t_{n+1}\mu_{-1}(u^{n+1} + v^{n+1})uv + \\ \mu_{-1}\mu_{n+1}(u^{n+2} + v^{n+2})(u+v) + \dots = 0 \end{aligned}$$

where  $\dots$  denotes the terms containing only  $t_m$  and  $\mu_m$  with  $m < n+1$ .

The coefficients of  $u^{n+2}v$  and  $u^{n+1}v^2$  have the form

$$\begin{aligned} (n+2)\mu_{n+1} + t_{n+1} + \dots &= 0, \\ (n+3)(n+2)\mu_{n+1} + \dots &= 0. \end{aligned}$$

The equations imply the Lemma.  $\square$

**Corollary 4.10.** *If  $t_{-1} \neq 0$ , then a solution of equation (4.16) is uniquely determined by the six parameters  $\mu_0(0), \mu_1(0), \mu_2(0), t_{-1}, t_1, t_3$ .*

Now we present a six parameter family of solutions. We shall use the solution of the CDYB equation (4.4) given in [FW],

$$r(\lambda, z, \tau) = -\rho(z, \tau) \sum_i x_i \otimes x_i - \sum_{\alpha \in \Delta} \sigma_{h_\alpha(\lambda)}(z, \tau) e_\alpha \otimes e_{-\alpha}$$

where the functions  $\rho$  and  $\sigma$  are defined in (4.6). For every  $\tau \in \mathbb{C}$ ,  $\text{Im } \tau > 0$ , the function  $r(\lambda, z, \tau)$  is a classical dynamical r-matrix with spectral parameter  $z$  and coupling constant  $\epsilon = -1$  [FW]. Hence, for every  $\tau$ , the functions  $t(z) = -\rho(z, \tau)$  and  $\mu(x, z) = -\sigma_x(z, \tau)$  form a solution of (4.16).

**Lemma 4.11.** *Let  $t(z)$  and  $\mu(x, z)$  be a solution of (4.16). Let  $A$  be a complex number, then*

$$\begin{aligned} t(Az), & \quad \mu(x, Az), \\ t(z), & \quad \mu(x + A, z), \\ At(z), & \quad A\mu(Ax, z), \\ t(z) + Az, & \quad e^{Azx} \mu(x, z), \\ t(z), & \quad e^{Az} \mu(x, z) \end{aligned}$$

are solutions of (4.16).

**Corollary 4.12.** *The functions*

$$(4.21) \quad \begin{aligned} t(z) &= -A\rho(Bz, \tau) + Dz, \\ \mu(x, z) &= -A\sigma_{Ax-C}(Bz, \tau)e^{z(Dx+E)} \end{aligned}$$

form a solution of (4.16) depending on six parameters  $A, B, C, D, E, \tau$ .

Let  $t, \mu$  be the solution of (4.16) defined by (4.21). Let  $t_{-1}, t_1, t_3, \mu_0(0), \mu_1(0), \mu_2(0)$  be the corresponding Taylor coefficients of  $t$  and  $\mu$ , cf. Corollary 4.10. These six Taylor coefficients are functions of the six parameters  $A, B, C, D, E, \tau$  and define a meromorphic map  $\chi : \mathbb{C}^5 \times H \rightarrow \mathbb{C}^6$ , where  $H$  is the upper half plane.

**Lemma 4.13.** *The Jacobian of this map is not identically equal to zero, and the image of the map is dense in  $\mathbb{C}^6$ .*

*Proof.* In order to prove that the Jacobian is not zero it suffices to show that all the six parameters  $A, \dots, \tau$  could be recovered from the function

$$\mu = -A \frac{\theta_1(Ax - C - Bz, \tau) \theta_1'(0, \tau)}{\theta_1(Ax - C, \tau) \theta_1(Bz, \tau)} e^{z(Dx+E)}.$$

In fact, the poles of this function are given by

$$Ax - C \in \mathbb{Z} \oplus \tau\mathbb{Z} \quad \text{and} \quad Bz \in \mathbb{Z} \oplus \tau\mathbb{Z}.$$

Knowing the poles we recover  $\tau, A, B, C$ . If  $Bz \mapsto Bz + 1$ , then  $\mu \mapsto \mu e^{(Dx+E)/B}$ , this property allows us to recover  $D, E$  at least locally.

Now let us show the density of the image of  $\chi$ . Let  $p(z) = \frac{d}{dz} \ln \mu(0, z) = -\frac{1}{z} + p_0 + p_1 z + p_2 z^2 + \dots$ . Consider the map  $\tilde{\chi} : \mathbb{C}^5 \times H \rightarrow \mathbb{C}^6$  given by the formula  $(A, B, D, E, C, \tau) \rightarrow (t_{-1}, t_1, t_3, p_0, p_1, p_2)$ . It is enough to show that the image of  $\tilde{\chi}$  is dense.

We have

$$t(z) = -A\rho(Bz, \tau) + Dz, \quad p(z) = E + \frac{d}{dz} \ln \frac{\theta_1(C + Bz, \tau)}{\theta_1(Bz, \tau)}.$$

Thus,  $t_1 = D + f_1(A, B, \tau)$ ,  $p_0 = E + f_2(B, C, \tau)$ , for some meromorphic functions  $f_1, f_2$ , and  $t_{-1}, t_3, p_1, p_2$  do not depend on  $D$  and  $E$ . Thus, to show the density of the image of  $\tilde{\chi}$ , it is enough to show the density of the image of  $\xi : \mathbb{C}^3 \times H \rightarrow \mathbb{C}^4$  given by  $(A, B, C, \tau) \rightarrow (t_{-1}, t_3, p_1, p_2)$ .

It is clear that the function  $p'(z)$  is doubly periodic with respect to  $C$ , with periods  $1, \tau$ . Therefore, for any fixed  $\tau$ , the map  $\xi$  is a *rational* map  $\xi_\tau : \mathbb{C}^2 \times E_\tau \rightarrow \mathbb{C}^4$ , where  $E_\tau$  is the elliptic curve corresponding to  $\tau$ . Denote by  $I_\tau$  the closure of the image of  $\xi_\tau$ . As the Jacobian of  $\xi$  is not identically zero, for generic  $\tau$ , the set  $I_\tau$  is an irreducible algebraic hypersurface in  $\mathbb{C}^4$ .

It is easy to see that each of the functions  $t''(z), p'(z)$  satisfies the modular invariance conditions

$$\begin{aligned} f(A, B, C, z, \tau + 2) &= f(A, B, C, z, \tau), \\ f(A, B, C, z, -1/\tau) &= f(A\tau, B\tau, C\tau, z, \tau). \end{aligned}$$

Therefore, the hypersurface  $I_\tau$  is modular invariant with respect to the subgroup  $\tilde{\Gamma}$  of the modular group  $\Gamma$  generated by  $\tau \rightarrow \tau + 2$ , and  $\tau \rightarrow -1/\tau$ . This means that the coefficients of the equation of  $I_\tau$  are modular functions on the modular curve  $\Sigma = H/\tilde{\Gamma}$  (it is easy to see that they have power growth in  $q$  as  $q = e^{2\pi i\tau} \rightarrow 0$ ). This shows that the hypersurfaces  $I_\tau$  form an algebraic family over  $\Sigma$ . Let  $T$  be the total space of this family. We have a natural rational map  $\psi : T \rightarrow \mathbb{C}^4$ , and the closure  $I$  of its image coincides with the closure of the image of  $\xi$ . Since the map is rational and has a nonzero Jacobian, we have  $I = \mathbb{C}^4$ , as desired.  $\square$

Corollaries 4.10, 4.12 and Lemma 4.13 tell us that all solutions  $t, \mu$  to equation (4.16) are limits of solutions given by (4.21). It is enough to list the limits of the function  $\mu$  since then the function  $t$  can be recovered from (4.16). Without loss of generality we assume that the coupling constant  $\epsilon$  is equal to 1, i.e.  $A = -B$ .

Let

$$f(x, z) = B \frac{\theta(B(x - x_0 - z), \tau) \theta'(0, \tau)}{\theta(B(x - x_0), \tau) \theta(Bz, \tau)} e^{z(Dx + E)}$$

be a function of  $x, z$  depending on parameters  $B, D, E, x_0, \tau$ . A function  $g(x, z)$  will be called a limit of the function  $f$  if there exist sequences  $B_n, D_n, E_n, x_{0,n}, \tau_n$  such that  $g(x, z)$  is the limit of  $f(x, z; B_n, D_n, E_n, x_{0,n}, \tau_n)$  when  $n$  tends to infinity.

**Proposition 4.14.**

*Any limit  $g$  of the function  $f$  has one of the following three forms.*

*Rational type.*

$$(4.22) \quad g = \frac{1}{z} e^{z(Dx + E)},$$

$$(4.23) \quad g = \frac{x - x_0 - z}{(x - x_0)z} e^{z(Dx + E)}$$

where  $D, E, x_0$  are parameters.

*Trigonometric type.*

$$(4.24) \quad g = \frac{2\pi B}{\sin(2\pi Bz)} e^{z(Dx+E)},$$

$$(4.25) \quad g = \frac{2\pi B \sin(2\pi B(x - x_0 - z))}{\sin(2\pi B(x - x_0)) \sin(2\pi Bz)} e^{z(Dx+E)}$$

where  $B, D, E, x_0$  are parameters.

*Elliptic type.*

$$(4.26) \quad g = B \frac{\theta(B(x - x_0 - z), \tau) \theta'(0, \tau)}{\theta(B(x - x_0), \tau) \theta(Bz, \tau)} e^{z(Dx+E)}$$

where  $B, D, E, x_0, \tau$  are parameters.

*Proof.* Let  $g$  be a limit of  $f$ . Introduce  $v_1 = \partial_z (\partial_x + \partial_z) \ln g$ ,  $v_2 = \partial_x (\partial_x + \partial_z) \ln g$ ,  $v_3 = \partial_x \partial_z \ln g$ .

**Lemma 4.15.** *The functions  $v_1, v_2, v_3$  have the following properties.*

1.  $v_1$  is a function of  $z$ ,  $v_2$  is a function of  $x$ ,  $v_3$  is a function of  $x - z$ .
2. After identifying the respective variables in these functions with a new variable  $t$ , we have  $v_2(t) = v_3(t)$ .
3.  $v_1(t) = t^{-2} + O(t^{-1})$ , if  $t \rightarrow 0$ .
4. The functions  $v_1, v_2, v_3$  satisfy a common differential equation of the form

$$(4.27) \quad (v')^2 = 4v^3 + pv^2 + qv + r$$

for suitable numbers  $p, q, r$ . Such an equation is unique.

*Proof.* Properties 1,2,3 are satisfied for  $g = f$  with any values of the parameters. Therefore, they are satisfied for any limit  $g$ .

Property 4 is satisfied for  $g = f$ . Using property 3, we conclude that for any limit  $g$  the function  $v_1$  satisfies a unique equation of form (4.27). Since for  $g = f$  the functions  $v_2, v_3$  satisfy the same equation as  $v_1$ , this is also true for any limit. The Lemma is proved.  $\square$

Let  $P(t)$  be the cubic polynomial on the right hand side of (4.27).

**Lemma 4.16.** *Let the roots of  $P$  be pairwise distinct. Then  $g$  has form (4.26).*

*Proof.* If the roots are distinct, then by Lemma 4.15 the function  $v_1(t)$  has the form  $B^2 \wp(Bt, \tau) + D$  where  $\wp$  is the Weierstrass function, and  $B, D, \tau$  are suitable constants. The functions  $v_2 = v_3$  satisfy the same differential equation (4.27) as  $v_1$ . Therefore, either  $v_2(t) = v_3(t) = v_1(t - t_0)$  for some  $t_0 \in \mathbb{C}$ , or  $v_2 = v_3 = \text{const}$ . It is clear that the second situation cannot arise in a limit of  $f$ , so  $v_2(t) = v_3(t) = v_1(t - t_0)$ .

If  $v_1, v_2, v_3$  are known, then the second differential of the logarithm of the function  $g$  is known, and hence the function  $g$  is known up to a transformation of the form  $g \mapsto g e^{ax+bz+c}$  for suitable constants  $a, b, c$ . Therefore,  $g = G e^{ax+bz+c}$  where  $G$  has form (4.26). The condition  $\text{Res}_{z=0} g = 1$  implies  $a = c = 0$ . Now  $g = G e^{bz}$  and the parameter  $b$  can be included into the parameter  $E$  of (4.26). The Lemma is proved.  $\square$



**Lemma 4.17.** *Let  $P$  have a root of multiplicity 2. Then  $g$  has form (4.24) or (4.25).*

*Proof.* An equation of the form  $(v')^2 = 4(v - \alpha)^2(v - \beta)$  can be solved explicitly. This gives

$$(4.28) \quad v_1(z) = \frac{4\pi^2 B^2}{\sin^2(2\pi Bz)} + D.$$

The function  $v_2 = v_3$  has to be a solution of this equation, so either  $v_2(t) = v_3(t) = v_1(t - t_0)$ , or  $v_2 = v_3 = \alpha$ , or  $v_2 = v_3 = \beta$ . It is easy to see that the third case cannot arise as a limiting case of  $f$ . In the first case,  $g$  has form (4.25) up to a factor  $e^{ax+bz+c}$  for suitable numbers  $a, b, c$ . Reasoning as before we conclude that  $g$  has form (4.25).

Similarly, in the second case,  $g$  has form (4.24). The Lemma is proved.  $\square$

**Lemma 4.18.** *If the polynomial  $P$  has a root of multiplicity 3, then  $g$  has form (4.22) or (4.23).*

*Proof.* Analogous to Lemma 4.17.  $\square$

The Proposition is proved.  $\square$

#### 4.5.5. End of proof of Theorem 4.2, part 2.

In the previous section we have determined the possible forms of the function  $\mu(x, z)$  for any root  $\alpha$ : they are given by (4.22)-(4.26) for coupling constant 1. Now we will determine the consistency conditions, which are imposed on these functions for different roots by the  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_\gamma$ -part of the CDYB equation, where  $\alpha + \beta + \gamma = 0$ .

First of all, by our assumptions the function  $\varphi_\alpha$  is a meromorphic function for any root  $\alpha$ . Since  $\varphi_\alpha = \mu e^{zL_\alpha\psi}$ , and  $\mu$  is meromorphic, the function  $\psi(\lambda)$  is holomorphic on  $\mathfrak{h}^*$ . Therefore, by using gauge transformations of type 1 and 2, it is possible to reduce any  $r$ -matrix with spectral parameter and coupling constant 1 to the form in which its  $\mathfrak{h} \otimes \mathfrak{h}$ -part is  $T(z) \sum x_i \otimes x_i$ , where  $T(z) = \frac{1}{z} + T_1 z + O(z^3)$  is an odd, scalar-valued meromorphic function. We will call such  $r$ -matrices reduced, and from now on will work only with them.

For a reduced  $r$ -matrix  $\varphi_\alpha(\lambda, z) = \mu_\alpha^*((\alpha, \lambda), z)$ , where  $\mu_\alpha^*(x, z) = \mu_\alpha(x, z)e^{T_1 xz}$ , and  $\mu_\alpha$  is the function  $\mu$  introduced in Section 4.5.2. Observe that  $\mu_\alpha^*$  is a function from family (4.21). Let  $A_\alpha, B_\alpha, C_\alpha, D_\alpha, E_\alpha, \tau_\alpha$  be the parameters  $A, B, C, D, E, \tau$  determined from (4.21), for  $\mu = \mu_\alpha^*$ . Since the coupling constant is 1, we have  $A = -B$ .

**Lemma 4.19.** *All  $\mu_\alpha^*$  are of the same type (rational, trigonometric, or elliptic).*

*Proof.* From equation (4.16) we can find the function  $t(u)$ . We know that it is the same for all roots  $\alpha$ . It is easy to check that  $\mu_\alpha^*$  is of rational, trigonometric, or elliptic type iff the set of poles of  $t(z)$  is a lattice of rank 0, 1, 2 respectively. The Lemma is proved.  $\square$

So it remains to consider the rational, trigonometric, and elliptic cases separately.

**Elliptic case.**

**Lemma 4.20.** *Let  $r$  be a reduced dynamical  $r$ -matrix such that the functions  $\mu_\alpha^*$  are of elliptic type. Then there exist complex numbers  $a, b, \tau$ ,  $\text{Im } \tau > 0$ , and elements  $\nu, \kappa \in \mathfrak{h}^*$  such that*

$$A_\alpha = a, \quad B_\alpha = b, \quad \tau_\alpha = \tau, \quad C_\alpha = (\alpha, \nu), \quad D_\alpha = c, \quad E_\alpha = (\alpha, \kappa),$$

for all roots  $\alpha$ .

*Proof.* Substitute formula (4.21) into the  $\mathfrak{g}_\alpha \otimes \mathfrak{g}_\beta \otimes \mathfrak{g}_\gamma$ -part of the CDYB equation, see (4.17). Considering the poles at the hyperplanes of the form  $z_{1,2} = \text{const}$ , we observe that the functions  $\varphi_\beta$  and  $\varphi_\gamma$  have the same lattice of periods. This allows us to conclude that there exist numbers  $b$  and  $\tau$  such that  $B_\alpha = b$ , and  $\tau_\alpha = \tau$  for all  $\alpha$ . The residue condition (4.3) implies the existence of a number  $a$  such that  $A_\alpha = a$  for all  $\alpha$ . The necessity to cancel the poles at the hyperplanes of the form  $(\alpha, \lambda) = \text{const}$  implies the existence of elements  $\kappa, \nu \in \mathfrak{h}^*$ ,  $c \in \mathbb{C}$  such that  $C_\alpha = (\alpha, \nu)$ ,  $D_\alpha = c$ ,  $E_\alpha = (\alpha, \kappa)$  for all  $\alpha$ .

This proves the Lemma, and Theorem 4.2, part 2, in the elliptic case.  $\square$

Now consider the trigonometric and rational case.

**Lemma 4.21.**  *$D_\alpha$  are the same for all  $\alpha$ .*

*Proof.* Easily follows from (4.17).  $\square$

Thus, we can reduce to the situation  $D_\alpha = 0$  by using a gauge transformation of type 2 with  $\psi = -D(\lambda, \lambda)/2$ .

**Rational case.**

**Lemma 4.22.** *In the rational case  $E_{\alpha+\beta} = E_\alpha + E_\beta$ .*

*Proof.* Follows directly from (4.17).  $\square$

Thus, in the rational case we can reduce to the situation  $E_\alpha = 0$  by a gauge transformation of type 2 with  $\psi = -(E, \lambda)$ ,  $E \in \mathfrak{h}^*$ .

If  $r$  is a reduced rational dynamical  $r$ -matrix with  $D_\alpha = E_\alpha = 0$ , then it is easy to see from Proposition 4.14 that  $r$  is of the form  $\frac{\Omega}{z} + r_0(\lambda)$ , where  $r_0$  is skew-symmetric. Since  $r_0 = \lim_{z \rightarrow \infty} r$ ,  $r_0(\lambda)$  is a classical dynamical  $r$ -matrix without spectral parameter with zero coupling constant. Such  $r$ -matrices were classified in Theorem 3.2. Theorem 3.2. implies that  $r(\lambda, z)$  is equivalent to (4.9) by gauge transformations of type 3. This proves Theorem 4.2, part 2, in the rational case.

**Trigonometric case.** In the trigonometric case, the functions  $\mu_\alpha^*$  have form (4.24), (4.25). As in the elliptic case, it is easy to see that  $B$  is the same for all  $\alpha$  since  $B^{-1}$  is the period of the lattice of poles. So by a gauge transformation of type 4 we can arrange  $B = 1$ .

**Lemma 4.23.** *Let  $\rho$  be the half-sum of positive roots. There exists a limit of  $r(s\rho, z)$  as  $s \rightarrow i\infty$*

*Proof.* The statement is clear from formulae (4.24), (4.25).

Denote by  $\bar{r}(z)$  this limit. This is a classical  $r$ -matrix with spectral parameter having the form  $\frac{\Omega}{z} + O(1)$  as  $z \rightarrow 0$  and invariant under the action of the Cartan subalgebra. A classification of such  $r$ -matrices was given by Belavin and Drinfeld [BD1], Theorem 6.1.

Namely, consider an r-matrix

$$(4.29) \quad r_{tr}(z) = 2i \frac{\Omega_- e^{2iz} + \Omega_+}{e^{2iz} - 1},$$

where  $\Omega_{\pm} = \frac{1}{2} \sum x_i \otimes x_i + \sum_{\alpha \in \Delta_{\pm}} e_{\alpha} \otimes e_{-\alpha}$  are the half-Casimirs. According to [BD1], any classical r-matrix of the above type can be obtained from  $r_{tr}(z)$  by change of polarization of the Lie algebra, and by gauge transformations of type 4, and type 2 with a linear function  $\psi$ .

Thus, in order to prove Theorem 4.2, part 2, in the trigonometric case, it is enough to assume that  $\bar{r}(z) = r_{tr}(z)$ .

Under this assumption, it is easy to deduce from Proposition 4.14 that  $\mu_{\alpha}^*(x, z)$  equals to:

1.  $\frac{\sin(x-x_0+z)}{\sin(x-x_0)\sin z}$  if  $\mu_{\alpha}^*$  depends nontrivially on  $x$ ;
2.  $\frac{e^{-iz}}{\sin z}$ , if  $\alpha > 0$  and  $\mu_{\alpha}^*$  does not depend on  $x$ .
3.  $\frac{e^{iz}}{\sin z}$ , if  $\alpha < 0$  and  $\mu_{\alpha}^*$  does not depend on  $x$ .

Now let us send  $z \rightarrow i\infty$ . It is easy to see that the limit of  $r(\lambda, z)$  exists. Denote this limit by  $r_{\infty}(\lambda)$ . Let  $\mu_{\alpha}^{\infty}((\lambda, \alpha))$  be the coefficient of  $e_{\alpha} \otimes e_{-\alpha}$  in  $r_{\infty}(\lambda)$ . From cases 1-3 above, we get that  $\mu_{\alpha}^{\infty}(x)$  equals to:

- 1'.  $\frac{e^{-i(x-x_0)}}{\sin(x-x_0)}$ , if  $\mu_{\alpha}^{\infty}$  depends nontrivially on  $x$ ;
- 2'.  $-2i$ , if  $\alpha > 0$  and  $\mu_{\alpha}^{\infty}$  does not depend on  $x$ .
- 3'.  $0$ , if  $\alpha < 0$  and  $\mu_{\alpha}^{\infty}$  does not depend on  $x$ .

On the other hand,  $r_{\infty}(\lambda)$  is a classical dynamical r-matrix without spectral parameter, and it is clear from cases 1'-3' that it has coupling constant  $-2i$ .

So,  $r_{\infty}(\lambda)$  has to be of the form given by Theorem 3.10. This determines possible combinations of functions  $\mu_{\alpha}^*$ , and shows that  $r(\lambda, z)$  is equivalent to an r-matrix (4.8) for a suitable subset  $X$  by a gauge transformation of type 3.

Theorem 4.2, part 2 is proved.

#### 4.6. Dynamical r-matrices without spectral parameter for affine Lie algebras.

In this section we interpret dynamical classical r-matrices without spectral parameter for an affine Lie algebra as dynamical r-matrices with spectral parameter for the underlying simple Lie algebra.

Let  $\mathfrak{g}$  be a simple Lie algebra, and  $\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$  be the corresponding affine Lie algebra, where  $c$  is the central element, and  $d$  is the grading element. Let  $\mathfrak{h} \subset \mathfrak{g}$  be a Cartan subalgebra, and  $\{x_i\}$  an orthonormal basis of  $\mathfrak{h}$ . Let  $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}$  be the Cartan subalgebra of  $\tilde{\mathfrak{g}}$ ,  $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ . Recall that  $c, d$  are orthogonal to  $\mathfrak{h}$  with respect to the standard bilinear form, and  $(c, d) = 1$ ,  $(c, c) = (d, d) = 0$ .

The elements of  $\tilde{\mathfrak{h}}$  have the form  $h + xc + yd$  where  $h \in \mathfrak{h}$ ,  $x, y \in \mathbb{C}$ . The elements of  $\tilde{\mathfrak{h}}^*$  are triples  $(\lambda, k, s)$  such that  $\langle (\lambda, k, s), h + xc + yd \rangle = \langle \lambda, h \rangle + kx + sy$ .

Let  $\delta \in \tilde{\mathfrak{h}}^*$  be the positive imaginary root,  $\langle \delta, d \rangle = 1$ ,  $\langle \delta, c \rangle = 0$ ,  $\langle \delta, \mathfrak{h} \rangle = 0$ . The roots of  $\tilde{\mathfrak{g}}$  are  $\alpha + n\delta$  and  $n\delta$ , where  $\alpha$  is a root of  $\mathfrak{g}$ ,  $n \in \mathbb{Z}$ , and in the second case  $n \neq 0$ . The positive roots are  $\alpha + n\delta$  and  $(n+1)\delta$ , where  $\alpha$  is a positive root of  $\mathfrak{g}$ , and  $n \geq 0$ .

A basis of positive root elements of  $\tilde{\mathfrak{g}}$  is formed by  $e_{\alpha}t^n$ ,  $e_{-\alpha}t^{n+1}$ , and  $x_it^{n+1}$ , where  $\alpha > 0$ ,  $n \geq 0$ . The dual elements are respectively  $e_{-\alpha}t^{-n}$ ,  $e_{\alpha}t^{-n-1}$ , and  $x_it^{-n-1}$ , where  $\alpha > 0$ ,  $n \geq 0$ .

According to Theorem 3.1, the solution of CDYBE for  $\tilde{\mathfrak{g}}$  with coupling constant 2 and  $C = 0, \nu = 0$  has the form

$$(4.23) \quad r(\lambda, k, s) = \hat{\Omega} + \sum_{\alpha \in \Delta, n} \cotanh((\alpha, \lambda) + kn) e_{\alpha} t^n \otimes e_{-\alpha} t^{-n} + \sum_i \sum_{n \neq 0} \cotanh(kn) x_i t^n \otimes x_i t^{-n}.$$

where  $\hat{\Omega}$  is the Casimir element of  $\tilde{\mathfrak{g}}$ .

For any  $z \in \mathbb{C}^*$ , let  $\pi_z : \mathfrak{g}[t, t^{-1}] \rightarrow \mathfrak{g}$  be the evaluation map at  $t = z$ . Consider the function  $\bar{r}(y, w) = (\pi_y \otimes \pi_w)(r|_{c=0})$ , where  $r|_{c=0} \in \mathfrak{g}[t, t^{-1}]^{\hat{\otimes} 2}$  is the image of  $r$  under the reduction modulo  $c$ . It is easy to see that  $\bar{r}$  depends only on  $u = y/w$ , and equals

$$(4.24) \quad \bar{r}(\lambda, k, u) = \sum_{\alpha \in \Delta} \left[ \sum_{n \in \mathbb{Z}} u^n (1 + \cotanh((\alpha, \lambda) + kn)) \right] e_{\alpha} \otimes e_{-\alpha} + \sum_i \left[ 1 + \sum_{n \neq 0} u^n (1 + \cotanh(kn)) \right] x_i \otimes x_i,$$

Set  $u = e^{2\pi i z}$ .

**Lemma 4.24.** *For any  $k$ , the function  $\hat{r}(\lambda, z) = \bar{r}(\lambda, k, u)$  satisfies the CDYB equation with spectral parameter  $z$ .*

*Proof.* Applying the operator  $\pi_{z_1} \otimes \pi_{z_2} \otimes \pi_{z_3}$  to the CDYB equation (3.3) for  $\tilde{\mathfrak{g}}$ , one easily obtains the CDYB equation with spectral parameter.  $\square$

Now we compute  $\hat{r}(\lambda, z)$ . Set  $\tau = k/\pi i$ , assume that  $\text{Im } \tau > 0$ , and use the classical formulae

$$(4.30) \quad \sum_{n \in \mathbb{Z}} u^n (1 + \cotanh(a + \pi i \tau n)) = -\frac{1}{\pi i} \sigma_{\frac{a}{\pi i}}(z, \tau),$$

and

$$(4.31) \quad 1 + \sum_{n \in \mathbb{Z} \setminus 0} u^n (1 + \cotanh(\pi i \tau n)) = -\frac{1}{\pi i} \rho(z, \tau),$$

where  $\sigma$  and  $\rho$  are defined in (4.6). Then  $\hat{r}(\lambda, z)$  takes the form of the Felder solution of CDYBE with spectral parameter,

$$\hat{r}(\lambda, z) = -\frac{1}{\pi i} \rho(z, \tau) \sum_{i=1}^N x_i \otimes x_i - \frac{1}{\pi i} \sum_{\alpha \in \Delta} \sigma_{\frac{(\alpha, \lambda)}{\pi i}}(z, \tau) e_{\alpha} \otimes e_{-\alpha}.$$

## Appendix: Open problems.

In conclusion we would like to formulate two open problems.

Let  $\mathfrak{g}$  be a simple Lie algebra,  $\mathfrak{h}$  its Cartan subalgebra, and  $\mathfrak{h}_0$  a subspace in  $\mathfrak{h}$ . We will say that a meromorphic function  $r : \mathfrak{h}_0^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  is a classical dynamical r-matrix if it satisfies (3.2), (3.3), and (3.1) for  $h \in \mathfrak{h}_0$ . We will say that a meromorphic function  $r : \mathfrak{h}_0^* \times \mathbb{C} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  is a classical dynamical r-matrix with a spectral parameter if it satisfies (4.2)-(4.4), and (4.1) for  $h \in \mathfrak{h}_0$ . The number  $\epsilon$  in both cases is called the coupling constant.

**Problem 1.** *Classify classical dynamical  $r$ -matrices on  $\mathfrak{h}_0$ , with a nonzero coupling constant.*

**Problem 2.** *Classify classical dynamical  $r$ -matrices on  $\mathfrak{h}_0$  with a spectral parameter, with a nonzero coupling constant.*

These problems are solved for two extreme cases –  $\mathfrak{h}_0 = 0$  ([BD1,BD2]) and  $\mathfrak{h}_0 = \mathfrak{h}$  (this paper). We expect that it can be solved in the intermediate cases by combining of the methods of our paper and the two papers of Belavin and Drinfeld.

**Note added in the second version:** While this paper was being revised, O.Schiffmann obtained a partial solution of Problem 1 [Sch].

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